Hille-Yosida Theorem and some Applications

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Contents

Introduction

The main focus of this work is going to be the Hille-Yosida theorem, which, as we will see, is a very powerful tool in solving evolution partial differential equations. The material is organized as follows. Chapter 1 presents the background material needed for the treatment in the subsequent chapters. In Chapter 2, we present the Hille-Yosida Theorem and related results and offer detailed proofs. Finally, in Chapter 3, we investigate applications of the Hille-Yosida theorem to some real word phenomena. We prove existence, uniqueness and regularity of solutions of the Heat equation, the Wave equation, and the linearized equations of coupled sound and heat flow. For the proofs of most of the results established in Chapter 1, relevant references are cited wherever needed. The subsequent chapters are self-contained.

Chapter 1

Preliminaries

This chapter is going to be our toolbox of important concepts and results that we will use time and again throughout the course of this Thesis.

1.1 Maximal monotone operators and their properties

Here we will only consider single-valued operators but to account for the general case of multi-valued operators, a good way to define an operator A on a set X is to simply consider A as a subset of the Cartesian product $X \times X$. Then we can define the domain of A as

$$D(A) = \{u \in X | \exists v \text{ such that } [u, v] \in A\}.$$

We use the notation $[u, v]$ to denote an element of the Cartesian product so as not to confuse with a scalar product which we will denote by $(\cdot, \cdot)$. Note that for a single valued operator A, $D(A) = \{u \in X | \exists v \in X \text{ such that } Au = v\}$. We also define the range of A as

$$R(A) = \bigcup_{u \in D(A)} Au.$$

For the case of single valued operators Au can be considered as the singleton set $\{Au\}$.

Henceforth, we will only focus on operators defined on a real Hilbert space, H.

Definition. An operator A on a Hilbert space H is called monotone if

$$(y_2 - y_1, x_2 - x_1) \geq 0 \quad \forall\ [x_1, y_1], [x_2, y_2] \in A.$$

Note that for a single valued linear operator A, it is monontone if $(Au, u) \geq 0\ \forall u \in D(A)$.

Definition. An operator A on a Hilbert space H is called maximal monotone if:

(i) A is monotone.

(ii) A has no proper monotone extension. That is, for any monotone operator $A' \subset H \times H$ if $A \subseteq A'$, then $A = A'$.

Theorem 1.1.1 (G. Minty). *Suppose $A : D(A) \subset H \to H$ is a monotone operator. Then A is maximal monotone iff $R(A + I) = H$.*

This is a very important result used in identifying maximal monotone operators. In fact, sometimes, this is taken as the definition of maximal operators.

Proposition 1.1.2. Let A be a linear maximal monotone operator. Then we have:

(i) $D(A)$ is dense in H;

(ii) A is a closed operator;

(iii) $(I + \lambda A)$ is a bijection from $D(A)$ onto H for every $\lambda > 0$. Moreover, $(I + \lambda A)^{-1}$ is a bounded operator and $\left\|(I + \lambda A)^{-1}\right\|_{\mathcal{L}(H)} \leq 1$.

Remark 1.1.1 We see from Theorem 1.1.1 that the following are equivalent:
(i) A is a maximal monotone operator.
(ii) $\lambda A + I$ is surjective for all $\lambda > 0$.
(iii) $\lambda A + I$ is surjective for some $\lambda > 0$.
That is, A is maximal monotone iff λA is maximal monotone for some $\lambda > 0$; iff λA is maximal montone for all $\lambda > 0$.

It also holds that $A + \lambda I$ is surjective for some some $\lambda > 0$ if and only if A is maximal monotone.

Definition. Let A be a maximal monotone operator. For every $\lambda > 0$, we define the following operators:

$$J_\lambda = (I + \lambda A)^{-1}, \qquad A_\lambda = \frac{1}{\lambda}(I - J_\lambda).$$

J_λ is called the resolvent, and A_λ, the Yosida approximation of A. Note that $D(J_\lambda) = D(A_\lambda) = H$, $R(J_\lambda) = D(A)$ and if A is linear, then $\|J_\lambda\|_{\mathcal{L}(H)} \leq 1$.

The resolvent and Yosida approximation (named after Kosaku Yosida) have many useful properties that make them an invaluable tool in proving the Hille-Yosida Theorem (Theorem 2.1.3). We list some of their properties as follows.

Proposition 1.1.3. Let A be a linear maximal monotone operator. Then we have the following:
(i) $A_\lambda v = A(J_\lambda v) \qquad \forall\, v \in H,\ \forall \lambda > 0,$
(ii) $A_\lambda v = J_\lambda(Av) \qquad \forall\, v \in D(A),\ \forall \lambda > 0,$
(iii) $|A_\lambda v| \leq |Av| \qquad \forall\, v \in D(A), \forall\ \lambda > 0,$
(iv) $\lim_{\lambda \to 0} J_\lambda v = v \qquad \forall\, v \in H,$
(v) $\lim_{\lambda \to 0} A_\lambda v = Av \qquad \forall\, v \in D(A),$
(vi) $(A_\lambda v, v) \geq 0 \qquad \forall v \in H, \forall\ \lambda > 0,$
(vii) $|A_\lambda v| \leq \frac{1}{\lambda}|v| \qquad \forall v \in H, \forall\ \lambda > 0.$

For further reference regarding the material of this section, see [2], [1].

1.2 L^p spaces

Let us denote $\mathbb{R} = (-\infty, \infty)$; $\mathbb{N} = \{0, 1, 2, \cdots\}$. Let X be a real real Banach space with the norm $\|\cdot\|_X$ and $\Omega \subset \mathbb{R}^N$ for some integer $N \geq 1$ be a Lebesgue measurable set.

Definition. Let $1 \leq p < \infty$. Then we define $L^p(\Omega; X)$ as the space of all equivalence classes of (strongly) measurable functions $f : \Omega \to X$ such that $x \mapsto \|f(x)\|_X^p$ is Lebesgue integrable and the equivalence relation is equality a.e on Ω.

In general when we write, $u \in L^p(\Omega; X)$, we mean u is a representative of the equivalence class of functions that agree with u a.e on Ω. $L^p(\Omega; X)$ is a real Banach space witht he norm

$$\|u\|_{L^p(\Omega;X)} = \left(\int_\Omega \|u(x)\|_X^p \right)^{\frac{1}{p}}.$$

For $p = \infty$, we define $L^\infty(\Omega; X)$ as the space of equivalence classes of measurable functions $f : \Omega \to X$ such that $x \mapsto \|f(x)\|_X$ is essentially bounded. $L^\infty(\Omega; X)$ is a real Banach space with the norm

$$\|u\|_{L^\infty(\Omega;X)} = \operatorname*{ess\,sup}_{x\in\Omega} \|u(x)\|_X \, .$$

When $X = \mathbb{R}$, we use the notation $L^p(\Omega)$. If $\Omega = (a, b) \subset \mathbb{R}$, we write $L^p(a, b; X)$. For more background material regarding this section , see [2].

1.3 Sobolev spaces

In this section, we will define Sobolev spaces. First, we establish some notations. Here, we assume Ω is a non-empty open subset of $\mathbb{R}^N$. We denote by $C^k(\Omega)$ the space of functions that are continuous on Ω and have continuous partial derivatives up to order k. We also define:

$$C^\infty(\Omega) = \{\phi \in C(\Omega) | \phi \text{ has continuous partial derivatives of any order}\} ,$$
$$C_c^\infty(\Omega) = \{\phi \in C^\infty(\Omega) | \text{supp } \phi \text{ is compact in } \Omega\} ,$$

where supp $\phi = \overline{\{x \in \Omega \mid \phi(x) \neq 0\}}$. Let $1 \leq p \leq \infty$.

Definition. The Sobolev space $W^{1,p}(\Omega)$ is defined as

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \middle| \begin{array}{l} \exists g_1, g_2, ..., g_N \in L^p(\Omega) \text{ such that} \\ \int_\Omega u \frac{\partial \varphi}{\partial x_i} = - \int_\Omega g_i \varphi, \ \forall \varphi \in C_c^\infty(\Omega), \ \forall i = 1, \cdots, N \end{array} \right\}$$

For $u \in W^{1,p}(\Omega)$, we define $\frac{\partial u}{\partial x_i} = g_i$. This definition makes sense since g_i is unique a.e (this follows from the fact that $\int_\Omega f\varphi = 0 \ \forall \varphi \in C_c^\infty(\Omega) \ \Rightarrow \ f = 0$ a.e on Ω). Here $\frac{\partial u}{\partial x_i}$ denotes the weak derivative.

For $p = 2$, we write, $H^1(\Omega) = W^{1,2}(\Omega)$. The space $W^{1,p}(\Omega)$ equipped with the norm

$$\|u\|_{W^{1,p}} = \left(\|u\|_{L^p}^p + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p}^p \right)^{\frac{1}{p}} \text{ for } 1 \leq p < \infty$$

and

$$\|u\|_{W^{1,\infty}} = \|u\|_{L^\infty} + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^\infty} \text{ for } p = \infty$$

is a real Banach space. The space $H^1(\Omega)$ is equipped with the scalar product

$$\begin{aligned}(u,v)_{H^1} &= (u,v)_{L^2} + \sum_{i=1}^{N}\left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right)_{L^2} \\ &= \int_\Omega uv + \sum_{i=1}^{N}\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} \\ &= \int_\Omega uv + \int_\Omega \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} \\ &= \int_\Omega uv + \int_\Omega \nabla u \cdot \nabla v.\end{aligned}$$

$H^1(\Omega)$ is a real Hilbert space with this scalar product and the associated norm is

$$\|u\|_{H^1} = \left(\int_\Omega u^2 + \int_\Omega |\nabla u|^2\right)^{\frac{1}{2}}.$$

Remark 1.3.1. If $u \in C^1(\Omega) \cap L^p(\Omega)$, and if $\frac{\partial u}{\partial x_i} \in L^p(\Omega)$ for all $i = 1, \cdots, N$ (here $\frac{\partial u}{\partial x_i}$ denotes the partial derivative in the usual sense), then $u \in W^{1,p}(\Omega)$ and the usual partial derivatives coincide with their Sobolev counterparts. So our notations are consistent.

Definition. For integers $m \geq 2$ and $1 \leq p \leq \infty$, we define the Sobolev space $W^{m,p}(\Omega)$ inductively as follows:

$$W^{m,p}(\Omega) = \left\{ u \in W^{m-1,p}(\Omega) \,\middle|\, \frac{\partial u}{\partial x_i} \in W^{m-1,p}(\Omega)\ , \forall i = 1, \cdots, N \right\}.$$

Equivalently, we can define:

$$W^{m,p}(\Omega) = \left\{ u \in L^p(\Omega) \,\middle|\, \begin{array}{l} \forall \alpha, \text{ with } |\alpha| \leq m, \exists g_\alpha \in L^p(\Omega) \text{ such that} \\ \int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega g_\alpha \varphi,\ \forall \varphi \in C_c^\infty(\Omega). \end{array} \right\},$$

where we use the standard multi-index notation $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$ with $\alpha_i \geq 0$, an integer and $|\alpha| = \sum_{i=1}^N \alpha_i$ and

$$D^\alpha \varphi = \frac{\partial^{|\alpha|}\varphi}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}.$$

We denote $D^\alpha u = g_\alpha$, and as before, this is well-defined. The space $W^{m,p}(\Omega)$ equipped with the norm

$$\|u\|_{W^{m,p}} = \left(\sum_{|\alpha|\leq m} \|D^\alpha u\|_{L^p}^p\right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty$$

and

$$\|u\|_{W^{m,p}} = \sum_{|\alpha| \le m} \|D^\alpha u\|_{L^\infty} \text{ for } p = \infty$$

is a real Banach space. We write $H^m(\Omega) = W^{m,2}(\Omega)$. The space $H^m(\Omega)$ equipped with the scalar product

$$(u, v)_{H^m} = \sum_{|\alpha| \le m} (D^\alpha u, D^\alpha v)_{L^2}$$

is a real Hilbert space.

Definition. Let $1 \le p \le \infty$. The Sobolev space $W_0^{m.p}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in $W^{m,p}(\Omega)$. More precisely, $u \in W_0^{m,p}(\Omega)$ iff $\exists$ a sequence of functions , $u_n \in C_c^\infty(\Omega)$ such that $\|u - u_n\|_{W^{m,p}} \to 0$.

As before, we write $H_0^m(\Omega) = W_0^{m,2}(\Omega)$.

Remark 1.3.2 Consider $\Omega = \mathbb{R}_+^N$. Then $\Gamma = \partial\Omega = \mathbb{R}^{N-1} \times \{0\}$. In this case it can be shown that the map $u \mapsto u|_\Gamma$ defined from $C_c^\infty(\mathbb{R}^N)$ into $L^p(\Gamma)$ extends by density to a bounded linear operator from $W^{1,p}(\Omega)$ into $L^p(\Gamma)$. The operator is defined to be the trace of u on Γ and it is also denoted by $u|_\Gamma$.

Now if Ω is a regular open set in $\mathbb{R}^N$, e.g., if Ω is of class C^1 with $\Gamma = \partial\Omega$ bounded, then it is possible to define the trace of a function $u \in W^{1,p}(\Omega)$ on $\Gamma = \partial\Omega$. In this case, $u|_\Gamma \in L^p(\Gamma)$ (for the surface measure $d\sigma$).

One important result regarding the trace is as follow: the kernel of the trace operator is $W_0^{1,p}(\Omega)$, that is,

$$W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega) \mid \; u|_\Gamma = 0 \right\}.$$

For further reference, see [11], [2].

1.4 Open sets of class C^m

First we denote the following sets:

$$R_+^N = \{x = (x_1, ..., x_N) \in \mathbb{R}^N \mid x_N > 0\},$$
$$Q = \{x = (x_1, ..., x_N) \in \mathbb{R}^N \mid (\sum_{i=1}^{N-1} x_i^2)^{1/2} < 1 \ , \ |x_N| < 1\},$$
$$Q_+ = R_+^N \cap Q,$$
$$Q_0 = \{(x_1, ..., x_{N-1}, 0) \in \mathbb{R}^N \mid (\sum_{i=1}^{N-1} x_i^2)^{1/2} < 1\}.$$

Definition. An open set Ω is said to be of class C^1 if for every $x \in \Gamma = \partial\Omega$, there exists a neighborhood U_x of x in $\mathbb{R}^N$ and a bijective map $H : Q \to U_x$ such that

$$H \in C^1(\overline{Q}),$$
$$H^{-1} \in C^1(\overline{U_x}),$$
$$H(Q_+) = U_x \cap Q,$$
$$H(Q_0) = U_x \cap \Gamma.$$

Definition. Similarly, an open set Ω is said to be of class C^m for an integer $m \geq 1$ if for every $x \in \Gamma = \partial\Omega$, there exists a neighborhood U_x of x in $\mathbb{R}^N$ and a bijection $H : Q \to U_x$ such that

$$H \in C^m(\overline{Q}),$$
$$H^{-1} \in C^m(\overline{U_x}),$$
$$H(Q_+) = U_x \cap \Omega,$$
$$H(Q_0) = U_x \cap \Gamma.$$

Ω is said to be of class C^∞ if it is of class C^m $\forall m \in \mathbb{N}$.

Remark 1.4.1 For an open set $\Omega \subset \mathbb{R}^N$, we say that the boundary $\Gamma = \partial\Omega$ is C^k, if for every $x \in \Gamma$, $\exists r > 0$ and a C^k function $\gamma : \mathbb{R}^{N-1} \to \mathbb{R}$ such that we have (after possibly relabeling and reorienting the coordinate axes)

$$\Omega \cap B(x, r) = \{x \in B(x, r) | x_N > \gamma(x_1, \cdots, x_{N-1})\}.$$

Likewise Γ is class C^∞, if it is C^k, $\forall k \in \mathbb{N}$.

This definition for Γ to be of class C^k and that for an open set, Ω of class C^k are equivalent. See [5] for further reference.

1.5 Sobolev embedding theorems

In this section we will state some useful results regarding continuous injections of Sobolev spaces to some L^p spaces. We will need these results later on. We suppose here that $\Omega \subseteq \mathbb{R}^N$ is an open set of class C^1 and $\Gamma = \partial\Omega$ is bounded, or else that $\Omega = \mathbb{R}^N_+$.

Theorem 1.5.1. *Let $1 \leq p < \infty$. Then the following hold:*

$$\begin{aligned} &W^{1,p}(\Omega) \subset L^q(\Omega), \quad \text{where } \tfrac{1}{q} = \tfrac{1}{p} - \tfrac{1}{N} \quad \text{if } p < N, \\ &W^{1,p}(\Omega) \subset L^q(\Omega), \quad \forall\, q \in [p, +\infty) \qquad \text{if } p = N, \\ &W^{1,p}(\Omega) \subset L^\infty(\Omega), \qquad\qquad\qquad \text{if } p > N, \end{aligned}$$

where all the above injections are continuous. Furthermore, if $p > N$, we have, $\forall\, u \in W^{1,p}(\Omega)$,

$$|u(x) - u(y)| \leq C \|u\|_{W^{1,p}} |x - y|^\alpha \quad \text{for a.e } x, y \in \Omega$$

where $\alpha = 1 - \frac{N}{p}$ and C depends only on Ω, p and N. In particular, $W^{1,p}(\Omega) \subset C(\overline{\Omega})$(this inclusion is modulo the choice of a continuous representative).

Theorem 1.5.2. *Let $m \geq 1$ be an integer and $1 \leq p < \infty$. Then the following hold:*

$$\begin{aligned} &W^{m,p}(\Omega) \subset L^q(\Omega), \quad \text{where } \tfrac{1}{q} = \tfrac{1}{p} - \tfrac{m}{N} \quad \text{if } \tfrac{1}{p} - \tfrac{m}{N} > 0, \\ &W^{m,p}(\Omega) \subset L^q(\Omega), \quad \forall\, q \in [p, +\infty) \qquad \text{if } \tfrac{1}{p} - \tfrac{m}{N} = 0, \\ &W^{m,p}(\Omega) \subset L^\infty(\Omega), \qquad\qquad\qquad \text{if } \tfrac{1}{p} - \tfrac{m}{N} < 0, \end{aligned}$$

where all the above injections are continuous. Furthermore, if $m - \frac{N}{p} > 0$ is not an integer, set $k = \left[m - \frac{N}{p}\right]$ and $\theta = m - \frac{N}{p} - k$ $(0 < \theta < 1)$. Then $\forall u \in W^{m,p}(\Omega)$,

$$\|D^\alpha u\|_{L^\infty} \leq C \|u\|_{W^{m,p}} \quad \forall\ \alpha \text{ with } |\alpha| \leq k$$

and

$$|D^\alpha u(x) - D^\alpha u(y)| \leq C \|u\|_{W^{m,p}} |x - y|^\theta \text{ for a.e } x, y \in \mathbb{R}^N, \forall \alpha \text{ with } |\alpha| = k.$$

In particular, if $m - N/p > 0$ is not an integer, then

$$W^{m,p}(\Omega) \subset C^k(\overline{\Omega}), \text{ where } k = \left[m - \frac{N}{p}\right]$$

and $C^k(\overline{\Omega}) = \left\{u \in C^k(\Omega) \mid D^\alpha u \text{ has a continuous extension on } \overline{\Omega},\ \forall \alpha \text{ with } |\alpha| \leq k\right\}$.

Proofs. (c.f. [2] , [5]) ■

Corollary 1.5.3. *Given any $k \in \mathbb{N}$, there exists an integer $m > k$ such that $W^{m,2}(\Omega) \subset C^k(\overline{\Omega})$ and this inclusion is a continuous injection.*

Proof. if $\Omega \subset \mathbb{R}^N$, where N is odd, then for sufficiently large $m > \frac{N}{2}$, we have that $m - \frac{N}{2} > 0$ is not an integer and it follows from Theorem 1.5.2 that $W^{m,2}(\Omega) \subset C^k(\overline{\Omega})$.

Now if N is even, choose $m_1 = \frac{N}{2}$, an integer. Then it follows from Theorem 1.5.2, that

$$W^{m_1,2}(\Omega) \subset L^q(\Omega) \text{ is a continuous injection} \tag{1.5.1}$$

for some irrational $q > 2$. Now let's choose an integer $m_2 = \left[k + \frac{N}{q}\right] + 1$. Then $m_2 - \frac{N}{q} > 0$ is not an integer and by Theorem 1.5.2 with $k = \left[m_2 - \frac{N}{q}\right]$,

$$W^{m_2,q}(\Omega) \subset C^k(\overline{\Omega}) \tag{1.5.2}$$

is a continuous injection. Now, since $W^{m_1,2}(\Omega) \subset L^q(\Omega)$,

$$u \in W^{m_1+1,2}(\Omega) = \left\{ u \in W^{m_1,2}(\Omega) \mid \frac{\partial u}{\partial x_i} \in W^{m_1,2}(\Omega),\ \forall\, i = 1, ..., N \right\}$$
$$\Rightarrow u \in L^q(\Omega), \frac{\partial u}{\partial x_i} \in L^q(\Omega),\ \forall\, i = 1, ..., N.$$

Therefore, $u \in W^{1,q}(\Omega)$, that is, $W^{m_1+1,2} \subset W^{1,q}(\Omega)$. Also note that $W^{m_1+1,2}(\Omega) \subset W^{m_1,2}(\Omega) \subset L^q(\Omega)$, where each injection is continuous. Now,

$$\|u\|^2_{W^{m_1+1,2}} = \|u\|^2_{W^{m_1,2}} + \sum_{|\alpha|=m_1+1} \|D^\alpha u\|^2_{L^2} \tag{1.5.3}$$
$$= \|u\|^2_{L^2} + \sum_{i=1}^{N} \sum_{|\alpha| \le m_1} \left\| D^\alpha \left(\frac{\partial u}{\partial x_i} \right) \right\|^2_{L^2}$$
$$= \|u\|^2_{L^2} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|^2_{W^{m_1,2}} \tag{1.5.4}$$

Therefore,

$$\|u\|_{W^{m_1+1,2}} \to 0$$
$$\Rightarrow \|u\|_{W^{m_1,2}} \to 0 \qquad \text{(from (1.5.3))}$$
$$\Rightarrow \|u\|_{L^q} \to 0 \qquad \text{(from (1.5.1))}$$

Again,

$$\begin{aligned} &\|u\|_{W^{m_1+1,2}} \to 0 \\ \Rightarrow &\left\|\frac{\partial u}{\partial x_i}\right\|_{W^{m_1,2}} \to 0 \qquad \text{(from (1.5.4))} \\ \Rightarrow &\left\|\frac{\partial u}{\partial x_i}\right\|_{L^q} \to 0 \qquad \text{(from (1.5.1))} \end{aligned}$$

for $i = 1$ to N. But $\|u\|_{L^q} \to 0$ and $\left\|\frac{\partial u}{\partial x_i}\right\|_{L^q} \to 0 \ \forall \ i = 1$ to N together imply $\|u\|_{W^{1,q}} \to 0$. Thus, $W^{m_1+1,2}(\Omega) \subset W^{1,q}(\Omega)$ is a continuous injection.

Proceeding inductively, and using a similar argument as above, we can show that $W^{m_1+l,2}(\Omega) \subset W^{l,q}(\Omega)$ for any integer $l \geq 1$. Then taking $l = m_2$, we have, $W^{m_1+m_2,q}(\Omega) \subset W^{m_2,q}(\Omega)$ is a continuous injection. And from (1.5.2), it follows that

$$W^{m_1+m_2,2}(\Omega) \subset C^k(\overline{\Omega})$$

is a continuous injection.Thus given any $k \in \mathbb{N}$, if we take $m \geq m_1 + m_2 = \frac{N}{2} + \left[k + \frac{N}{q}\right] + 1$,

$$W^{m,2}(\Omega) \subset C^k(\overline{\Omega})$$

is a continuous injection.

■

1.6 Green's identity for Sobolev Spaces

We define the gradient and the Laplacian for Sobolev functions as follows:

$$\nabla u = \text{grad } u = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_N}\right),$$
$$\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2},$$

where all the partial derivatives are in the Sobolev sense. For example, $\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} = \sum_{i=1}^{N} g_i$, where

$$\int_\Omega u \frac{\partial^2 \varphi}{\partial x_i^2} = \int_\Omega g_i \varphi \quad \forall \ \varphi \in C_c^\infty(\Omega).$$

Clearly, ∇u makes sense for $u \in H^1(\Omega)$ and Δu makes sense for $u \in H^2(\Omega)$. If in addition $u \in C^1(\Omega)$, by Remark 1.3.1, ∇u (in the usual classical) sense coincides with its Sobolev counterpart. It's important to note here that Sobolev functions are considered equal if they agree a.e. Similarly, if $u \in C^2(\Omega)$ then Δu (in the usual classical sense) coincides with its Sobolev counterpart. So our notations are consistent. Also note that both ∇ and Δ are linear operators.

Green's identity for Sobolev functions is stated as follows:

Theorem 1.6.1. *For any $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, we have*

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega v\Delta u = \int_{\partial\Omega} v(\nabla u \cdot \vec{n})\mathrm{d}\sigma \tag{1.6.1}$$

where $\vec{n}$ is the outward pointing unit normal on the surface element $\mathrm{d}\sigma$*.*

Proof. We know that (1.6.1) holds for $u \in C^2(\Omega)$ and $v \in C^1(\Omega)$ in which case $\nabla u, \nabla v, \Delta u$ coincide with their usual classical counterparts. This identity can be extended to Sobolev spaces if both sides are continuous wrt the Sobolev norm. We will make use of the fact that if two continuous functions agree on a dense subset, then they agree everywhere.

Note that for linear expressions like $u \mapsto \int_\Omega \varphi \nabla u$ or bilinear like $(u, v) \mapsto \int_\Omega u \nabla v$, continuity and boundedness are equivalent. Note that

$$\begin{aligned} \left| \int_\Omega v\Delta u \right| &\leq \|v\|_{L^2} \|\Delta u\|_{L^2} && \text{(by C-S inequality)} \\ &\leq C \|v\|_{H^1} \|u\|_{H^2} . \end{aligned}$$

Similarly, it can be shown that $\int_\Omega \nabla u \cdot \nabla v$ is bounded. Thus the left hand side of 1.6.1 is continuous on $H^2(\Omega) \times H^1(\Omega)$. For a regular open set $\Omega \subset \mathbb{R}^N$(for example, Ω is of class C^1 and $\Gamma = \partial\Omega$ is bounded), the trace operator is bounded from $H^1(\Omega) \to L^2(\Gamma)$, and thus,

$$\begin{aligned} \left| \int_\Gamma v(\nabla u \cdot \vec{n})\mathrm{d}\sigma \right| &\leq \|v\|_{L^2(\Gamma)} \|\nabla u\|_{L^2(\Gamma)} \\ &\leq C_1 \|v\|_{H^1} \|u\|_{H^2} . \end{aligned}$$

That takes care of the RHS. ■

Note that in the case when $v \in H_0^1(\Omega)$ and $u \in H^2(\Omega)$, Green's identity reads

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega v\Delta u = 0 \tag{1.6.2}$$

because $v \in H_0^1(\Omega) \Rightarrow v|_\Gamma = 0$.

1.7 Variational formulation of the Dirichlet boundary value problem for the Laplacian

In this section we set up the variational formulation of the Dirichlet boundary value problem for the Laplacian and state some important results that we will use time and again later. Let $\Omega \subseteq \mathbb{R}^N$ be an open set. We are looking for a solution $u : \overline{\Omega} \to \mathbb{R}$ of the problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma = \partial\Omega \end{cases} \tag{1.7.1}$$

for a given function f. The condition $u = 0$ on Γ is called the (homogeneous) Dirichlet condition. Here $\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$.

A **classical solution** of the above problem is a function $u \in C^2(\overline{\Omega})$ that satisfies (1.7.1). A **weak solution** of the problem is defined to be a function $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega uv = \int_\Omega fv \quad \forall\ v \in H_0^1(\Omega).$$

Note that if $u \in H_0^1(\Omega)$ then $u|_\Gamma = 0$ and the boundary condition is incorporated in the definition.

It can be shown that a classical solution is also a weak solution. We have the following theorem that deals with the existence and the uniqueness of a weak solution.

Theorem 1.7.1 (Dirichlet, Riemann, Poincare-Hilbert)**.** *Given any $f \in L^2(\Omega)$, $\exists$ a unique weak solution $u \in H_0^1(\Omega)$ of (1.7.1). The unique solution is given by*

$$\min_{v \in H_0^1(\Omega)} \left\{ \frac{1}{2} \int_\Omega (|\nabla u|^2 + |v|^2) - \int_\Omega fv \right\}.$$

Proof. (cf. [2], [5]) ■

It can be further shown that if the weak solution $u \in H_0^1(\Omega)$ is also in $C^2(\overline{\Omega})$ and Ω is of class C^1, then the weak solution actually turns out to be a classical solution.

(*A similar treatment can be done for the Neumann boundary problem*

$$\begin{cases} -\Delta u + u = f & \text{in } \Gamma, \\ \dfrac{\partial u}{\partial n} = 0, \end{cases}$$

where $\frac{\partial u}{\partial n} = \nabla u \cdot \vec{n}$*,* $\vec{n}$ *being the outward pointing unit normal vector to* Γ*).*

The next theorem deals with the regularity of the weak solution for the Dirichlet problem.

Theorem 1.7.2. *Suppose* Ω *is of class* C^2 *and* $\Gamma = \partial\Omega$ *is bounded or else* $\Omega = \mathbb{R}^N_+$*. Let* $f \in L^2(\Omega)$ *and* $u \in H^1_0(\Omega)$ *satisfy*

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega u\varphi = \int_\Omega f\varphi \quad \forall\ \varphi \in H^1_0(\Omega).$$

Then $u \in H^2(\Omega)$ *and* $\|u\|_{H^2} \leq C\|f\|_{L^2}$ *where the constant* C *depends only on* Ω*. Moreover, if* Ω *is of class* C^{m+2} *and* $f \in H^m(\Omega)$*, then* $u \in H^{m+2}(\Omega)$ *and* $\|u\|_{H^{m+2}} \leq C\|f\|_{H^m}$*.*

Furthermore, if $m > \frac{N}{2}$ *and* $f \in H^m(\Omega)$*, then* $u \in C^2(\overline{\Omega})$*. And if* Ω *is of class* C^∞*, and* $f \in C^\infty(\overline{\Omega})$*, then* $u \in C^\infty(\overline{\Omega})$*.*

Proof. (cf. [2]) ■

Corollary 1.7.3. *Consider the unbounded linear operator* $A : D(A) \subset H \to H$*, where*

$$\begin{cases} H = L^2(\Omega), \\ D(A) = H^2(\Omega) \cap H^1_0(\Omega), \\ Au = -\Delta u. \end{cases}$$

Then A *is a maximal montone operator.(Here* $\Delta u = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}$ *in the Sobolev sense.)*

Proof. Note that by Theorem 1.7.1 and Theorem 1.7.2, it follows that given any $f \in L^2(\Omega)$, $\exists$ a unique $u \in H^2(\Omega) \cap H^1_0(\Omega)$ such that $-\Delta u + u = f$, that is, $(A + I)u = f$. So, $R(A + I) = H = L^2(\Omega)$. A is also monotone and therefore, by Minty's Theorem A is maximal monotone. ■

Remark 1.7.1. We saw that $-\Delta : H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega)$ *is a maximal monotone operator. Then by Remark 1.1.1, we have that* $-\Delta + \lambda I$ *is a bijection from* $H^2(\Omega) \cap$

$H_0^1(\Omega)$ onto $L^2(\Omega)$, $\forall\, \lambda > 0$. Then for any $f \in L^2(\Omega), \exists$ a unique $u \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $(-\Delta + \lambda I)u = f$, that is, $-\Delta u + \lambda u = f$.

Remark 1.7.2. On the other hand, suppose $u \in H^2(\Omega) \cap H_0^1(\Omega)$. Then $\exists$ a unique $f = f_u \in L^2(\Omega)$ such that $-\Delta u + u = f_u$. And by Theorem 1.7.2, we have

$$\|u\|_{H^2} \leq C \|f_u\|_{L^2}$$

for some constant C that depends only on Ω. Putting $f_u = -\Delta u + u$, we have

$$\|u\|_{H^2}^2 \leq C^2 \|-\Delta u + u\|_{L^2}^2 \leq 2C^2 \left(\|\Delta u\|_{L^2}^2 + \|u\|_{L^2}^2 \right).$$

Furthermore if, $u \in H^{k+2}(\Omega)$, we have $\Delta u \in H^k(\Omega)$. In this case, $f_u = -\Delta u + u \in H^k(\Omega)$. Suppose Ω is of class C^{k+2}. Then by Theorem 1.7.2, we have $\|u\|_{H^{k+2}} \leq C \|f_u\|_{H^k}$. Therefore, putting $f_u = -\Delta u + u$,

$$\|u\|_{H^{k+2}}^2 \leq 2C^2 \left(\|\Delta u\|_{H^k}^2 + \|u\|_{H^k}^2 \right).$$

Chapter 2

Hille-Yosida Theorem

In this chapter, we will state and prove the central theorem of the thesis, viz. the Hille-Yosida Theorem, named after the mathematicians Einar Hille and Kosaku Yosida who independently discovered the result around 1948. We will establish some results first along our way to the proof. Many of the results will be applied in the final chapter. We will adapt the proofs presented in [2] and offer detailed explanations wherever necessary.

2.1 Existence and uniqueness of solution to the evolution problem $\frac{\mathrm{d}u}{\mathrm{d}t} + Au = 0$ on $[0, +\infty)$ with initial data $u(0) = u_0$

We begin with the following general result:

Theorem 2.1.1 (Cauchy-Lipschitz-Picard)**.** *Let E be a Banach space. Let $F : E \to E$ be a Lipschitz map with Lipschitz constant L, that is,*

$$\|F(u) - F(v))\| \leq L \|u - v\| \;\; \forall u, v \in E$$

Then for every $u_0 \in E$, there exists a unique solution, u to the problem:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = F(u(t)), & \text{on } [0, +\infty) \\ u(0) = u_0 \end{cases} \tag{2.0.1}$$

such that $u \in C^1([0, +\infty); E)$.

Proof. Note that, finding a solution $u \in C^1([0, +\infty); E)$ to the above problem is equivalent to finding a solution $u \in C([0, +\infty); E)$ of the following equation:

$$u(t) = u_0 + \int_0^t F(u(s))ds, \;\; t \geq 0. \tag{2.0.2}$$

First we will define an appropriate Banach space of functions, X. Let

$$X = \left\{ u \in C([0,+\infty); E) \mid \sup_{t\in[0,+\infty)} e^{-tk} \|u(t)\| < \infty \right\}$$

where k is a positive constant that we will fix later. We will now check that X is indeed a Banach space for the norm

$$\|u\|_X = \sup_{t\in[0,+\infty)} e^{-tk} \|u(t)\|$$

Consider any Cauchy sequence $(u_n) \subseteq X$. That means, given any $\epsilon > 0$, $\exists N_\epsilon$ such that $\forall m, n > N_\epsilon$,

$$\|u_n - u_m\|_X < \epsilon.$$

For some fixed t, take $\epsilon e^{-tk} > 0$. Since $\{u_n\}$ is a Cauchy sequence, $\exists N = N_{\epsilon e^{-tk}}$ such that $\forall m, n > N$,

$$\begin{aligned} & \|u_n - u_m\|_X < \epsilon e^{-tk} \\ \Rightarrow & \sup_{t\geq 0} e^{-tk} \|u_n(t) - u_m(t)\| < \epsilon e^{-tk} \\ \Rightarrow & \ e^{-tk} \|u_n(t) - u_m(t)\| < \epsilon e^{-tk}, \forall t \geq 0 \\ \Rightarrow & \|u_n(t) - u_m(t)\| < \epsilon, \forall t \geq 0. \end{aligned}$$

This holds for any $\epsilon > 0$. That is, $\{u_n(t)\}$ is a Cauchy sequence in E. Since E is a Banach space, this means $\{u_n(t)\}$ converges to some point in E. Let's denote it by $u(t)$ That is,

$$u(t) := \lim_{n\to\infty} u_n(t).$$

Now we show that the function u defined as above is in X. Since $\{u_n(t)\}$ converges to $u(t)$, there exists N_1 such that for all $n > N_1$,

$$\|u(t) - u_n(t)\| < \epsilon/3.$$

Similarly, since $\{u_n(t_0)\}$ converges to $u(t_0)$, there exists N_2 such that for all $n > N_2$,

$$\|u(t_0) - u_n(t_0)\| < \epsilon/3.$$

Note that u_n is continuous. Therefore, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|u_n(t) - u_n(t_0)\| < \epsilon/3.$$

whenever $|t - t_0| < \delta$. Now take $n > max(N_1, N_2)$. Then, whenever $|t - t_0| < \delta$, we have,

$$\begin{aligned}
&\|u(t) - u(t_0)\| \\
&\leq \|u(t) - u_n(t) + u_n(t) - u_n(t_0) + u_n(t_0) - u(t_0)\| \\
&\leq \|u(t) - u_n(t)\| + \|u_n(t) - u_n(t_0)\| + \|u_n(t_0) - u(t_0)\| \\
&\leq \epsilon/3 + \epsilon/3 + \epsilon/3 \\
&= \epsilon.
\end{aligned}$$

Thus we showed that $u : [0, +\infty) \to E$ is continuous, that is, $u \in C([0, +\infty); E)$.

Now, for a fixed t, and for any $\epsilon > 0$, choose n large enough such that $\|u(t) - u_n(t)\| < \epsilon$. Note that $u_n \in X$ implies that $\sup_{t \geq 0} e^{-tk} \|u_n(t)\| < \infty$. That means $\exists M > 0$ such that $e^{-tk} \|u_n(t)\| < M, \forall\ t \geq 0$. That is,

$$\|u_n(t)\| < Me^{tk}.$$

Therefore,

$$\begin{aligned}
\|u(t)\| &= \|u(t) - u_n(t) + u_n(t)\| \\
&\leq \|u(t) - u_n(t)\| + \|u_n(t)\| \\
&< \epsilon + Me^{tk}\ , \forall \epsilon > 0.
\end{aligned}$$

This implies $\|u(t)\| < Me^{tk}$. The choice of t was arbitrary, so it holds for all $t \geq 0$. Thus, for all $t \geq 0$, $e^{-tk} \|u(t)\| < M$, that is, $\sup_{t \geq 0} e^{-tk} \|u(t)\| < \infty$ Thus, we have shown that $u \in X$.

Next we show that $\|u_n - u\|_X \to 0$. Since u_n is a Cauchy sequence in X, given $\epsilon > 0$, there exists N_ϵ such that $\forall\ m, n > N_\epsilon$,

$$\begin{aligned}
&\|u_n - u_m\|_X < \epsilon \\
&\Rightarrow \sup_{t \geq 0} e^{-tk} \|u_n(t) - u_m(t)\| < \epsilon \\
&\Rightarrow \|u_n(t) - u_m(t)\| < \epsilon e^{tk}, \forall\ t \geq 0.
\end{aligned}$$

Taking the limit as $m \to \infty$, we have

$$\begin{aligned} &\|u_n(t) - u(t)\| < \epsilon e^{tk}, \quad \forall\, t \geq 0\, \forall n > N_\epsilon \\ \Rightarrow &\sup_{t\geq 0} e^{-tk} \|u_n(t) - u(t)\| < \epsilon \quad \forall n > N_\epsilon \\ \Rightarrow &\|u_n - u\|_X < \epsilon \quad \forall n > N_\epsilon. \end{aligned}$$

That is,

$$\|u_n - u\|_X \to 0$$

This concludes the proof that X is Banach with the norm $\|\cdot\|_X$.

Now let us define a function $\Phi : X \to X$ by

$$(\Phi u)(t) = u_0 + \int_0^t F(u(s))ds$$

We claim that $\Phi u \in X$. Note that

$$\begin{aligned} \|(\Phi u)(t)\| &= \left\| u_0 + \int_0^t F(u(s))ds \right\| \\ &\leq \|u_0\| + \int_0^t \|F(u(s)) - F(u_0) + F(u_0)\|\, ds \\ &\leq \|u_0\| + \int_0^t \|F(u(s)) - F(u_0)\|\, ds + \int_0^t \|F(u_0)\|\, ds \\ &\leq \|u_0\| + L \int_0^t \|u(s) - u_0\|\, ds + t\,\|F(u_0)\| \\ &\leq \|u_0\| + Lt\,\|u_0\| + L \int_0^t \|u(s)\|\, ds + t\,\|F(u_0)\|\,. \end{aligned} \tag{2.0.3}$$

Now we derive some inequalities.

$$\begin{aligned} L \int_0^t \|u(s)\|\, ds &= L \int_0^t e^{-sk} e^{sk} \|u(s)\|\, ds \\ &\leq L\,\|u\|_X \int_0^t e^{sk} ds \\ &\quad (\text{ since } e^{-sk}\|u(s)\| \leq \sup_{t\geq 0} e^{-tk}\|u(t)\| = \|u\|_X \text{ }) \\ &\leq L\,\|u\|_X \frac{(e^{tk} - 1)}{k}. \end{aligned} \tag{2.0.4}$$

From (2.0.4) we can derive,

$$\begin{aligned}
e^{-tk}L\int_0^t \|u(s)\|\,ds &= e^{-tk}L\,\|u\|_X\,\frac{(e^{tk}-1)}{k} \\
&= L\,\|u\|_X\,\frac{(1-e^{tk})}{k} \\
&\le \frac{L}{k}\,\|u\|_X\,. && (2.0.5)
\end{aligned}$$

We also have,

$$e^{-tk}\,\|u_0\| \le \|u_0\|\,. \qquad (2.0.6)$$

Note that $e^{tk} > tk$. So, $e^{-tk} < 1/tk$. Therefore,

$$\begin{aligned}
e^{-tk}Lt\,\|u_0\| &\le \frac{1}{tk}Lt\,\|u_0\| \\
&= \frac{L}{k}\,\|u_0\|\,. && (2.0.7)
\end{aligned}$$

Also, we have

$$\begin{aligned}
e^{-tk}t\,\|F(u_0)\| &\le \frac{1}{tk}t\,\|F(u_0)\| \\
&= \frac{1}{k}\,\|F(u_0)\|\,. && (2.0.8)
\end{aligned}$$

Finally, combining (2.0.3), (2.0.5), (2.0.6), (2.0.7), (2.0.8), we get

$$\begin{aligned}
& e^{-tk}\,\|(\Phi u)(t)\| \le \|u_0\| + \frac{L}{k}\,\|u_0\| + \frac{L}{k}\,\|u\|_X + \frac{1}{k}\,\|F(u_0)\|\| && (\forall\ t \ge 0) \\
\Rightarrow & \sup_{t\ge 0} e^{-tk}\,\|(\Phi u)(t)\| < \infty. && (2.0.9)
\end{aligned}$$

Next we show that $(\Phi u) \in C([0,+\infty);E)$. Let $g(t) = \int_0^t F(u(s))ds$. Note that g(t) is continuous. That is,

$$\lim_{t\to t_0} \|g(t) - g(t_0)\| = 0. \qquad (\text{for any } t_0 \ge 0)$$

Therefore,

$$\begin{aligned}
& \lim_{t\to t_0} \|(\Phi u)(t) - (\Phi u)(t_0)\| \\
=\ & \lim_{t\to t_0} \|g(t) - g(t_0)\| \\
=\ & 0.
\end{aligned}$$

This together with (2.0.9) helps us conclude that $\Phi u \in X$.
Furthermore, we claim that

$$\|\Phi u - \Phi v\|_X \le \frac{L}{k}\,\|u - v\|_X\,.$$

Indeed, note that

$$\begin{aligned}
e^{-tk}\left\|(\Phi u)(t)-(\Phi v)(t)\right\| &= e^{-tk}\left\|\int_0^t (F(u(s)-F(v(s)))ds\right\| \\
&\leq e^{-tk}L\int_0^t \left\|u(s)-v(s)\right\| ds \\
&= e^{-tk}L\int_0^t e^{sk}e^{-sk}\left\|u(s)-v(s)\right\| ds \\
&\leq e^{-tk}L\left\|u-v\right\|_X \int_0^t e^{sk}ds \\
&= L\left\|u-v\right\|_X e^{-tk}\left[\frac{(e^{tk}-1)}{k}\right] \\
&= L\left\|u-v\right\|_X \frac{(1-e^{-tk})}{k} \\
&\leq \frac{L}{k}\left\|u-v\right\|_X .
\end{aligned}$$

Taking $\sup_{t\geq 0}$ of the left side, we obtain

$$\left\|\Phi u-\Phi v\right\|_X \leq \frac{L}{k}\left\|u-v\right\|_X . \tag{2.0.10}$$

Taking any $k > L$, we see from (2.0.10) that $\Phi : X \to X$ is a contraction map. Thus from Banach's Fixed Point Theorem, we get that Φ has a unique fixed point. Denote it by u. Then $u = \Phi u$. That is,

$$u(t) = (\Phi u)(t) = u_0 + \int_0^t F(u(s))\mathrm{d}s.$$

But then clearly, u is a solution of (2.0.1) with $u(0) = u_0$. This proves the existence part.

Now we prove uniqueness. Suppose u and $\bar{u}$ are two solutions to (2.0.1). Then using (2.0.2), we have,

$$\begin{aligned}
\left\|u(t)-\bar{u}(t)\right\| &= \left\|\int_0^t (F(u(s))-F(\bar{u}(s)))\, ds\right\| \\
&\leq \int_0^t \left\|(F(u(s))-F(\bar{u}(s)))\right\| ds \\
&\leq L\int_0^t \left\|(u(s)-\bar{u}(s))\right\| ds.
\end{aligned}$$

That is,

$$\phi(t) \leq L\int_0^t \phi(s)\ , \text{ where } \phi(t) = \left\|u(t)-\bar{u}(t)\right\| . \tag{2.0.11}$$

Let $f(t) = \int_0^t \phi(s)ds$. Then $f'(t) = \phi(t)$. Therefore, from (2.0.11), we have

$$\begin{aligned} &f'(t) \leq Lf(t) \\ \Rightarrow &f'(t) - Lf(t) \leq 0. \end{aligned} \tag{2.0.12}$$

Let $h(t) = e^{-Lt} f(t)$. Then

$$\begin{aligned} h'(t) &= e^{-Lt} f'(t) - Le^{-Lt} f(t) \\ &= e^{-Lt}(f'(t) - Lf(t)) \\ &\leq 0. \end{aligned}$$

Thus $h(t)$ is a non-increasing function. So we have

$$\begin{aligned} &h(t) \leq 0, \quad \text{for any } t \geq 0 \\ \Rightarrow\ &e^{-Lt} f(t) \leq 0. \end{aligned}$$

But $e^{-Lt} > 0$, so it must be that $f(t) \leq 0$. Again,

$$f(t) = \int_0^t \phi(s)ds = \int_0^t \left\| u(s) - \bar{u(s)} \right\| ds \geq 0.$$

Therefore, $f(t) = 0$ for all $t \geq 0$. That is,

$$\begin{aligned} &\phi(t) \leq Lf(t) = 0 \\ \Rightarrow\ &\phi(t) = 0,\ \forall t\ \geq 0 \\ \Rightarrow\ &\phi \equiv 0. \end{aligned}$$

Thus, we have shown uniqueness of the solution. This concludes our proof.

■

Now we will go on to state and prove the Hille-Yosida Theorem. Henceforth, by H, we will denote a real Hilbert space. First we will state and prove the following lemmas which we will use during the course of the proof.

Lemma 2.1.2. *Let $w \in C^1([0, +\infty); H)$ be a function satisfying*

$$\frac{\mathrm{d}w}{\mathrm{d}t} + A_\lambda w = 0 \text{ on } [0, +\infty) \tag{2.1.a}$$

where $A_\lambda = \frac{1}{\lambda}(I - J_\lambda)$ is the Yosida approximation, and J_λ is the resolvent of a maximal monotone operator A as defined in Section 1.1. Then the functions $t \mapsto |w(t)|$ and $t \mapsto |\frac{\mathrm{d}w}{\mathrm{d}t}(t)|$ are non-increasing on $[0, +\infty)$

Proof. First note that

$$\begin{aligned}
\frac{\mathrm{d}|w(t)|^2}{\mathrm{d}t} &= \lim_{h\to 0}\left[\frac{|w(t+h)|^2-|w(t)|^2}{h}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[(w(t+h),w(t+h))-(w(t),w(t))\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[(w(t+h),w(t+h))-(w(t),w(t+h))+(w(t+h),w(t))-(w(t),w(t))\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[(w(t+h)-w(t),w(t+h))+(w(t+h)-w(t),w(t))\right] \\
&= \lim_{h\to 0}\left(\frac{w(t+h)-w(t)}{h},w(t+h)\right)+\lim_{h\to 0}\left(\frac{w(t+h)-w(t)}{h},w(t)\right) \\
&= 2\left(\frac{\mathrm{d}w(t)}{\mathrm{d}t},w(t)\right). \qquad (2.1.b)
\end{aligned}$$

Since w satisfies (2.1.a), we have $\left(\frac{\mathrm{d}w(t)}{\mathrm{d}t},w(t)\right)=(-A_\lambda w(t),w(t))=-(A_\lambda w(t),w(t))$. Recall that $(A_\lambda v,v)\geq 0$ for all $v\in H$. Therefore,

$$\begin{aligned}
\frac{\mathrm{d}|w(t)|^2}{\mathrm{d}t} &= -(A_\lambda w(t),w(t)) \\
&\leq 0.
\end{aligned}$$

That is, $|w(t)|^2$ is an non-increasing function on $[0,+\infty)$. Then, $|w(t)|=(|w(t)|^2)^{1/2}$ is also non-increasing on $[0,+\infty)$.

Since A_λ is a linear bounded operator we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}w}{\mathrm{d}t}\right)+A_\lambda\left(\frac{\mathrm{d}w}{\mathrm{d}t}\right)=0,\ t\geq 0. \qquad (2.1.e)$$

So, $\mathrm{d}w/\mathrm{d}t$ satisfies (2.1.a). Using the same argument as that used in showing $|w(t)|$ is non-increasing, we conclude that $|\mathrm{d}w/\mathrm{d}t|$ is non-increasing.

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}^k w}{\mathrm{d}t^k}\right)+A_\lambda\left(\frac{\mathrm{d}^k w}{\mathrm{d}t^k}\right)=0,t\geq 0. \qquad \text{(for any order k)}$$

This is turn implies $|\frac{\mathrm{d}^k w}{\mathrm{d}t^k}|$ is non-increasing for any order k.

■

Lemma 2.1.3. *Consider a sequence of functions $f_n \in C^1([a,b];H)$. Suppose the sequence of functions $\frac{\mathrm{d}f_n}{\mathrm{d}t}$ converges uniformly to a function $g:[a,b]\to H$. If $f_n(t_0)$ converges for some point t_0 on $[a,b]$, then f_n converges uniformly to some function, f (say). Then f is differentiable and $\frac{\mathrm{d}f}{\mathrm{d}t} = g$.*

Proof. A similar result holds for real valued functions (see Theorem 7.17 in [16]).For our case, let's fix some $v \in H$. Denote $h_n(t) = (f_n(t), v)$; $h(t) = (f(t), v)$; $h_1(t) = (g(t), v)$. Note that $h'_n(t) = (f'_n(t), v)$ and these are all scalar functions. So we can apply Theorem 7.17 ([16]) to conclude that the sequence of function h_n converges uniformly to the function $h(t) = (f(t), v)$ and $h'(t) = h_1(t) = (g(t), v), \forall t \in [a,b]$. That is, $(f'(t), v) = (g(t), v)$. The choice of $v \in H$ was arbitrary. Therefore, $f'(t) = g(t)$.

■

Lemma 2.1.4. *Let $u_0 \in D(A)$. Then for any $\epsilon > 0$, $\exists \bar{u_0} \in D(A^2)$ such that $|u_0 - \bar{u_0}| < \epsilon$ and $|Au_0 - A\bar{u_0}| < \epsilon$. That is, $D(A^2)$ is dense in $D(A)$ for the graph norm.*

Proof. Take $u_0 \in D(A)$. Recall that $D(J_\lambda) = H, R(J_\lambda) = D(A)$. Denote $\bar{u_0} = J_\lambda u_0 = (I + \lambda A)^{-1} u_0$. Then $\bar{u_0} \in D(A)$ and $u_0 = \bar{u_0} + \lambda A \bar{u_0}$.

Therefore, $\lambda A\bar{u_0} = u_0 - \bar{u_0} \in D(A) \Rightarrow A\bar{u_0} \in D(A)$. But $\bar{u_0} \in D(A), A\bar{u_0} \in D(A)$ $\Rightarrow \bar{u_0} \in D(A^2)$.

Now recall that

$$J_\lambda(Av) = A_\lambda v \quad \forall v \in D(A), \forall\ \lambda > 0$$

and

$$A(J_\lambda v) = A_\lambda v \quad \forall v \in H, \forall\ \lambda > 0$$

Therefore, $\forall\ v \in D(A)$, $\forall\ \lambda > 0$, we have

$$J_\lambda(Av) = A(J_\lambda v) \tag{2.1.h}$$

Also, recall that $\lim_{\lambda\to 0} J_\lambda v = v$, $\forall\ v \in H$. Therefore,

$$\lim_{\lambda\to 0} |J_\lambda u_0 - u_0| = 0$$

and

$$\lim_{\lambda\to 0} |J_\lambda(Au_0) - Au_0| = 0$$

By the first inequality above , we have, for sufficiently small λ,

$$|\bar{u_0} - u_0| = |J_\lambda u_0 - u_0| < \epsilon \tag{2.1.i}$$

And by the second inequality. we have for sufficiently small λ,

$$\begin{aligned} &|J_\lambda(Au_0) - Au_0| < \epsilon \\ \Rightarrow &|A(J_\lambda u_0) - Au_0| < \epsilon && \text{(using (2.1.h))} \\ \Rightarrow &|A\bar{u_0} - Au_0| < \epsilon && \text{(2.1.j)} \end{aligned}$$

The equations (2.1.i) and (2.1.j) conclude the proof of the Lemma. ■

Theorem 2.1.5 (Hille-Yosida)**.** *Let $A : D(A) \subseteq H \to H$ be a maximal monotone operator, where H is a real Hilbert space. Then given any $u_0 \in D(A)$, there exists a unique solution, $u \in C^1([0, +\infty); H) \bigcap C([0, +\infty); D(A))$ to the following problem:*

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + Au = 0 \ \text{on} \ [0, +\infty) \\ u(0) = u_0. \end{cases} \tag{2.1.1}$$

Furthermore,

$$\begin{aligned} &|u(t)| \leq |u_0|, \\ &\left|\frac{\mathrm{d}u}{\mathrm{d}t}(t)\right| = |Au(t)| \leq |Au_0|, \quad \forall t \geq 0. \end{aligned}$$

Remark. Note that the space $D(A)$ is equipped with the norm $(|v|^2 + |Av^2|)^{1/2}$ or with the equivalent norm $|v| + |Av|$.

Proof. First we prove the uniqueness part.
Suppose u and $\bar{u}$ be two solutions to the system (2.1.1). Then, we have

$$\frac{\mathrm{d}(u - \bar{u})}{\mathrm{d}t} = -A(u - \bar{u}).$$

Therefore,

$$\begin{aligned} \left(\frac{\mathrm{d}(u - \bar{u})}{\mathrm{d}t}, u - \bar{u}\right) &= -\left(A(u - \bar{u}), (u - \bar{u})\right) \\ &\leq 0 \qquad \text{(since } A \text{ is monotone, } (Av, v) \geq 0 \text{ ,} \forall\ v \in H.) \end{aligned}$$

Let $\phi(t) = u(t) - \bar{u}(t)$. Then the above inequality reads

$$\left(\frac{\mathrm{d}\phi(t)}{\mathrm{d}t}, \phi(t)\right) \leq 0.$$

Note that $|\phi|^2 = (\phi, \phi)$. We have

$$\begin{aligned}
\frac{\mathrm{d}|\phi(t)|^2}{\mathrm{d}t} &= \lim_{h\to 0}\left[\frac{|\phi(t+h)|^2 - |\phi(t)|^2}{h}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[(\phi(t+h),\phi(t+h)) - (\phi(t),\phi(t))\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[(\phi(t+h),\phi(t+h)) - (\phi(t),\phi(t+h)) + (\phi(t+h),\phi(t)) - (\phi(t),\phi(t))\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[(\phi(t+h)-\phi(t),\phi(t+h)) + (\phi(t+h)-\phi(t),\phi(t))\right] \\
&= \lim_{h\to 0}\left(\frac{\phi(t+h)-\phi(t)}{h},\phi(t+h)\right) + \lim_{h\to 0}\left(\frac{\phi(t+h)-\phi(t)}{h},\phi(t)\right) \\
&= 2\left(\frac{\mathrm{d}\phi(t)}{\mathrm{d}t},\phi(t)\right) \qquad (2.1.\circledast) \\
&\leq 0.
\end{aligned}$$

This means $|\phi(t)|^2$ is an non-increasing function on $[0, +\infty)$. Then for any $t \geq 0$,

$$|\phi(t)|^2 \leq |\phi(0)|^2 = |u(0) - \bar{u}(0)|^2 = |u_0 - u_0|^2 = 0.$$

Therefore, $|\phi(t)| = 0$, for any $t \geq 0$. That is, $\phi \equiv 0$. This proves uniqueness.

Now we will prove the Existence part. We will use the following strategy. We replace A by A_λ in (2.1.1) and we use Theorem 2.1.1(Cauchy-Picard-Lipschitz) on the approximate problem. Then using the fact that $\lim_{\lambda\to 0} A_\lambda v = Av$ and a number of estimates that are independent of λ, we pass on to the limit as $\lambda \to 0^+$.

Recall that

$$|A_\lambda v| \leq \frac{1}{\lambda}|v| \quad \forall\ v \in H \text{ and } \forall \lambda > 0.$$

That is, A_λ is Lipschitz with Lipschitz constant $1/\lambda$.
Using Theorem 2.1.1, we have that for any $\lambda > 0$ there exists a solution (say u_λ) of the problem:

$$\begin{cases} \dfrac{\mathrm{d}w}{\mathrm{d}t}(t) = -A_\lambda w(t) \text{ on } [0, +\infty), \\ w(0) = u_0 \quad \in D(A), \end{cases}$$

such that $u_\lambda \in C^1([0,+\infty);H)$.

That is, for any $\lambda > 0$,

$$\begin{cases} \dfrac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t) = -A_\lambda u_\lambda(t) \text{ on } [0,+\infty), \\ u_\lambda(0) = u_0. \end{cases} \tag{2.1.2}$$

Now using Lemma 2.1.2, we have $|u_\lambda(t)|$ and $\left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)\right|$ are non-increasing on $[0,+\infty)$. Therefore, for any $t \geq 0$, $|u_\lambda(t)| \leq |u(0)| = |u_0|$ and

$$\begin{aligned} \left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)\right| &\leq \left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(0)\right| \\ \Rightarrow |A_\lambda u_\lambda(t)| &\leq |A_\lambda u_\lambda(0)| = |A_\lambda u_0| \leq |Au_0|. \end{aligned}$$

The last inequality holds because $|A_\lambda v| \leq |Av| \ \ \forall v \in D(A), \ \ \forall \lambda > 0$. So we have the following estimates:

$$|u_\lambda(t)| \leq |u_0| \ \ \forall\, t \geq 0, \ \ \forall \lambda > 0, \tag{2.1.3}$$

$$|A_\lambda u_\lambda(t)| \leq |Au_0| \ \ \forall\, t \geq 0, \ \ \forall \lambda > 0. \tag{2.1.4}$$

Now, note that for any $\lambda, \mu > 0$, we have

$$\begin{aligned} &\frac{\mathrm{d}u_\lambda}{\mathrm{d}t} - \frac{\mathrm{d}u_\mu}{\mathrm{d}t} + A_\lambda u_\lambda - A_\mu u_\mu = 0 \\ \Rightarrow\ &\frac{\mathrm{d}(u_\lambda - u_\mu)}{\mathrm{d}t} + A_\lambda u_\lambda - A_\mu u_\mu = 0 \\ \Rightarrow\ &\frac{\mathrm{d}(u_\lambda - u_\mu)}{\mathrm{d}t} = -(A_\lambda u_\lambda - A_\mu u_\mu). \end{aligned}$$

Using the same argument as (2.1.b) , we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_\lambda - u_\mu|^2 &= \left(\frac{\mathrm{d}(u_\lambda - u_\mu)}{\mathrm{d}t}, u_\lambda - u_\mu\right) \\ &= -(A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - u_\mu). \end{aligned} \tag{2.1.5}$$

Now,

$$\begin{aligned} &(A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - u_\mu) \\ = &(A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - J_\lambda u_\lambda + J_\lambda u_\lambda - J_\mu u_\mu + J_\mu u_\mu - u_\mu) \\ = &(A_\lambda u_\lambda - A_\mu u_\mu, (u_\lambda - J_\lambda u_\lambda) - (u_\mu - J_\mu u_\mu)) + (A_\lambda u_\lambda - A_\mu u_\mu, J_\lambda u_\lambda - J_\mu u_\mu). \end{aligned}$$

Recall that $A_\lambda = \frac{1}{\lambda}(I - J_\lambda)$. Therefore,

$$u_\lambda - J_\lambda u_\lambda = (I - J_\lambda)u_\lambda = \lambda A_\lambda u_\lambda.$$

Again recall that $A_\lambda v = A(J_\lambda v)\ \forall v \in H$ and $\forall \lambda > 0$. Then we have,

$$\begin{aligned}
&(A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - u_\mu)\\
&= (A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu) + (A(J_\lambda u_\lambda - J_\mu u_\mu), J_\lambda u_\lambda - J_\mu u_\mu)\\
&\geq (A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu). \qquad (2.1.6)
\end{aligned}$$

The last inequality holds because A is maximal monotone and hence $(Av, v) \geq 0\ \forall\, v \in D(A)$. From (2.1.5), we have,

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_\lambda - u_\mu|^2 &= -(A_\lambda u_\lambda - A_\mu u_\mu, u_\lambda - u_\mu)\\
&\leq -(A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu) \qquad \text{(from 2.1.6)}\\
&= -\lambda(A_\lambda u_\lambda, A_\lambda u_\lambda) + \mu(A_\lambda u_\lambda, A_\mu u_\mu) + \lambda(A_\mu u_\mu, A_\lambda u_\lambda) - \mu(A_\mu u_\mu, A_\mu u_\mu)\\
&= (\lambda + \mu)(A_\lambda u_\lambda, A_\mu u_\mu) - \lambda|A_\lambda u_\lambda|^2 - \mu|A_\mu u_\mu|^2\\
&\leq (\lambda + \mu)|(A_\lambda u_\lambda, A_\mu u_\mu)| + \lambda|A_\lambda u_\lambda|^2 + \mu|A_\mu u_\mu|^2\\
&\leq (\lambda + \mu)|A_\lambda u_\lambda| \cdot |A_\mu u_\mu| + \lambda|A_\lambda u_\lambda|^2 + \mu|A_\mu u_\mu|^2\\
&\qquad \text{(using Cauchy-Schwarz inequality)}\\
&\leq 2(\lambda + \mu)|Au_0|^2.
\end{aligned}$$

For the last inequality, we used the estimate (2.1.4). So we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_\lambda - u_\mu|^2 \leq 4(\lambda + \mu)|Au_0|^2.$$

Integrating the inequality with respect to t, we get

$$\begin{aligned}
&|u_\lambda - u_\mu|^2 \leq 4t(\lambda + \mu)|Au_0|^2\\
\Rightarrow\ &|u_\lambda - u_\mu| \leq 2\sqrt{t(\lambda + \mu)}. \qquad (\circledast)
\end{aligned}$$

It is obvious from $\circledast$ that $u_\lambda(t)$ converges uniformly as $\lambda \to 0^+$ on every bounded interval $[0, T]$. Note that each $u_\lambda(t) \in C([0 + \infty); H)$. It follows from the uniform limit theorem that the limit function $u \in C([0, +\infty); H)$.

By Lemma 2.1.2, if $u_\lambda \in C^1([0, +\infty); H)$ satisfies $\frac{\mathrm{d}w}{\mathrm{d}t} + A_\lambda w = 0$ on $[0, +\infty)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}\right) + A_\lambda\left(\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}\right) = 0,\ t \geq 0.$$

Also from the proof of Lemma 2.1.2, we have $u_\lambda \in C^\infty([0,+\infty);H)$. Let's denote $v_\lambda = \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}$, then $v_\lambda \in C^\infty([0,+\infty);H)$ and $\frac{\mathrm{d}v_\lambda}{\mathrm{d}t} + A_\lambda v_\lambda = 0$ Then

$$\frac{\mathrm{d}(v_\lambda - v_\mu)}{\mathrm{d}t} = -(A_\lambda v_\lambda - A_\mu v_\mu).$$

Using the same argument as (2.1.b), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v_\lambda - v_\mu|^2 &= \left(\frac{\mathrm{d}(v_\lambda - v_\mu)}{\mathrm{d}t}, v_\lambda - v_\mu\right)\\ &= -(A_\lambda v_\lambda - A_\mu v_\mu, v_\lambda - v_\mu).\end{aligned}$$

Now,

$$\begin{aligned}&(A_\lambda v_\lambda - A_\mu v_\mu, v_\lambda - v_\mu)\\ &= (A_\lambda v_\lambda - A_\mu v_\mu, v_\lambda - J_\lambda v_\lambda + J_\lambda v_\lambda - J_\mu v_\mu + J_\mu v_\mu - v_\mu)\\ &= (A_\lambda v_\lambda - A_\mu v_\mu, \lambda A_\lambda v_\lambda - \mu A_\mu v_\mu) + (A_\lambda v_\lambda - A_\mu v_\mu, J_\lambda v_\lambda - J_\mu v_\mu)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{since } A_\lambda = 1/\lambda(I - J_\lambda))\\ &= (A_\lambda v_\lambda - A_\mu v_\mu, \lambda A_\lambda v_\lambda - \mu A_\mu v_\mu) + (A(J_\lambda v_\lambda - J_\mu v_\mu), J_\lambda v_\lambda - J_\mu v_\mu)\\ &\geq (A_\lambda v_\lambda - A_\mu v_\mu, \lambda A_\lambda v_\lambda - \mu A_\mu v_\mu). \qquad (\text{since } (Av,v) \geq 0,\ A \text{ being monotone})\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v_\lambda - v_\mu|^2\\ &= -(A_\lambda v_\lambda - A_\mu v_\mu, \lambda A_\lambda v_\lambda - \mu A_\mu v_\mu)\\ &\leq |(A_\lambda v_\lambda - A_\mu v_\mu, \lambda A_\lambda v_\lambda - \mu A_\mu v_\mu)|\\ &\leq |A_\lambda v_\lambda - A_\mu v_\mu| \cdot |\lambda A_\lambda v_\lambda - \mu A_\mu v_\mu| \qquad (\text{by Cauchy-Schwarz inequality})\\ &\leq (|A_\lambda v_\lambda| + |A_\mu v_\mu|)(\lambda |A_\lambda v_\lambda| + \mu |A_\mu v_\mu|). \qquad (2.1.7)\end{aligned}$$

Again from the proof of Lemma 2.1.2, $\left|\frac{\mathrm{d}v_\lambda}{\mathrm{d}t}\right| = \left|\frac{\mathrm{d}^2 u_\lambda}{\mathrm{d}t^2}\right|$ is non-increasing on $[0,+\infty)$. Therefore,

$$\begin{aligned}\left|\frac{\mathrm{d}v_\lambda}{\mathrm{d}t}(t)\right| &\leq |A_\lambda v_\lambda(0)| = \left|A_\lambda \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(0)\right|\\ &= |A_\lambda A_\lambda u_0| = |A_\lambda^2 u_0|\\ \Rightarrow |A_\lambda v_\lambda(t)| &\leq |A_\lambda^2 u_0| \quad \forall\ \lambda > 0. \qquad (2.1.8)\end{aligned}$$

Now we assume that $u_0 \in D(A^2)$, so, $Au_0 \in D(A)$. Since $A_\lambda v = J_\lambda(Av)$, $\forall\ v \in D(A)$, $\forall\ \lambda > 0$, we have

$$A_\lambda^2 u_0 = A_\lambda(A_\lambda u_0) = A_\lambda(J_\lambda(Au_0)) = J_\lambda A(J_\lambda(Au_0)) = J_\lambda J_\lambda A(Au_0) = J_\lambda^2 A^2 u_0.$$

Therefore,

$$\begin{aligned}|A_\lambda^2 u_0| &= |J_\lambda^2 A^2 u_0| \le \|J_\lambda\|^2_{\mathcal{L}(H)} \, |A^2 u_0| \\ &\le |A^2 u_0|. \end{aligned} \tag{2.1.9}$$

The last inequality holds because $\|J_\lambda\|_{\mathcal{L}(H)} \le 1$ for all $\lambda > 0$.

From (2.1.7) and (2.1.8), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v_\lambda - v_\mu|^2 &\le 2|A_\lambda^2 u_0|(\lambda+\mu)|A_\lambda^2 u_0| \\ &= 2(\lambda+\mu)|A_\lambda^2 u_0|^2 \\ &\le 2(\lambda+\mu)|A^2 u_0|^2. \qquad \text{(from (2.1.9))}\end{aligned}$$

Integrating wrt t, we have

$$|v_\lambda(t) - v_\mu(t)| \le 2\sqrt{t(\lambda+\mu)}|A^2 u_0|. \tag{2.1.9a}$$

Therefore $v_\lambda \to v$ uniformly on every bounded interval $[0,T]$ as $\lambda \to 0^+$. Since $v_\lambda \in C([0,+\infty);H)$, v being the uniform limit of v_λ, satisfies $v \in C([0,T];H)$ for every $T > 0$, and hence $v \in C([0,+\infty);H)$.

So far we have, as $\lambda \to 0^+$

$$\begin{aligned}&u_\lambda(t) \to u(t) \text{ uniformly on } [0,T] \\ &\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t) \to v(t) \text{ uniformly on } [0,T], \quad \forall\, T > 0.\end{aligned} \tag{2.1.10}$$

By Lemma 2.1.3, we have u has a derivative $\frac{\mathrm{d}u}{\mathrm{d}t}$ and $v(t) = \frac{\mathrm{d}u}{\mathrm{d}t}(t)$. Besides , since $\frac{\mathrm{d}u_\lambda}{\mathrm{d}t} \in C([0,+\infty);H)$, and $v(t) = \frac{\mathrm{d}u}{\mathrm{d}t}(t)$ is the uniform limit, it also holds that $\frac{\mathrm{d}u}{\mathrm{d}t} \in C([0,+\infty);H)$. And hence, $u \in C^1([0,+\infty);H)$.

Now note that we can write $\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t) + A_\lambda u_\lambda(t) = 0$ as

$$\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t) + A(J_\lambda u_\lambda(t)) = 0. \qquad \text{(since } A_\lambda x = A(J_\lambda x) \quad \forall\, x \in H, \quad \forall\, \lambda > 0)$$

Recall that $\lim_{\lambda\to 0} J_\lambda x = x$. Therefore

$$\begin{aligned}
&|J_\lambda u_\lambda(t) - u(t)| \\
\le&|J_\lambda u_\lambda(t) - J_\lambda u(t)| + |J_\lambda u(t) - u(t)| \\
\le&\|J_\lambda\|_{\mathcal{L}(H)} |u_\lambda(t) - u(t)| + |J_\lambda u(t) - u(t)| \\
\to&0 \text{ as } \lambda \to 0^+
\end{aligned}$$

That is,

$$J_\lambda u_\lambda(t) \to u(t) \text{ as } \lambda \to 0^+. \tag{2.1.11}$$

Again, from (2.1.10) and (2.1.11),

$$A(J_\lambda u_\lambda(t)) = -\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t) \to -\frac{\mathrm{d}u}{\mathrm{d}t} \text{ as } \lambda \to 0^+.$$

Since A is closed, this means $u(t) \in D(A)$ and

$$\begin{aligned}
&-\frac{\mathrm{d}u}{\mathrm{d}t}(t) = Au(t) \\
\Rightarrow&\frac{\mathrm{d}u}{\mathrm{d}t}(t) + Au(t) = 0.
\end{aligned}$$

Furthermore, note that since $u \in C^1([0, +\infty); H)$, $Au(t) = -\frac{\mathrm{d}u}{\mathrm{d}t} \in C([0, +\infty); H)$. It follows that

$$\begin{aligned}
|u(t) - u(t_0)|_{D(A)} &= |u(t) - u(t_0)| + |Au(t) - Au(t_0)| \\
&\to 0 \text{ as } t \to t_0 \text{ for all } t_0 \ge 0
\end{aligned}$$

Thus, $u \in C([0, +\infty); D(A))$. Moreover, recall that from the derived estimates (2.1.3) and (2.1.4), we have

$$\begin{aligned}
|u_\lambda(t)| &\le |u_0| \quad \forall\, t \ge 0, \quad \forall \lambda > 0, \\
|A_\lambda u_\lambda(t)| &\le |Au_0| \quad \forall\, t \ge 0, \quad \forall \lambda > 0.
\end{aligned}$$

And passing on to the limit as $\lambda \to 0^+$, we have

$$\begin{aligned}
|u(t)| &\le |u_0| \quad \forall\, t \ge 0, \\
|Au(t)| &\le |Au_0| \quad \forall\, t \ge 0.
\end{aligned}$$

Now note that by Lemma 2.1.4, $D(A^2)$ is dense in $D(A)$. Therefore, for any $u_0 \in D(A)$, we can find a sequence $(u_{0n}) \subseteq D(A^2)$ such that $|u_{0n} - u_0|_{D(A)} \to 0$. That is,

$$|u_{0n} - u_0| + |Au_{0n} - Au_0| \to 0.$$

That is, $u_{0n} \to u_0$ and $Au_{0n} \to Au_0$. By the preceding analysis, for every $u_{0n} \in D(A^2)$, we know that the problem:

$$\begin{cases} \dfrac{\mathrm{d}w}{\mathrm{d}t} + Aw = 0 \text{ on } [0, +\infty), \\ w(0) = u_{0n}, \end{cases}$$

has a solution , say, u_n which satisfies

$$\begin{aligned} &|u_n(t)| \le |u_0| \\ &\left|\frac{\mathrm{d}u_n}{\mathrm{d}t}(t)\right| = |Au_n(t)| \le |Au_0| \quad \forall\, t \ge 0. \end{aligned}$$

Note that $u_n - u_m$ is a solution to the problem

$$\begin{cases} \dfrac{\mathrm{d}w}{\mathrm{d}t} + Aw = 0 \text{ on } [0, +\infty), \\ w(0) = u_{0n} - u_{0m}, \end{cases}$$

therefore, for all $t \ge 0$, we have

$$|u_n(t) - u_m(t)| \le |u_{0n} - u_{0m}|, \tag{2.1.12}$$

$$\left|\frac{\mathrm{d}u_n}{\mathrm{d}t}(t) - \frac{\mathrm{d}u_m}{\mathrm{d}t}(t)\right| \le |Au_{0n} - Au_{0m}|. \tag{2.1.13}$$

Since $u_{0n} \to u_0$ in H, (u_{0n}) is a Cauchy sequence in H. Therefore given any $\epsilon > 0$ for sufficiently large m, n, we have

$$|u_n(t) - u_m(t)| \le |u_{0m} - u_{0n}| < \epsilon$$

That is, $(u_n(t))$ is a Cauchy sequence in H. Thus $(u_n(t))$ has a pontwise limit in H. Let's denote it as $u(t) := \lim_{n\to\infty} u_n(t)$. Then from (2.1.12),

$$\begin{aligned} &\lim_{m\to\infty} |u_n(t) - u_m(t)| \le \lim_{m\to\infty} |u_{0n} - u_{0m}| \\ \Rightarrow &|u_n(t) - u(t)| \le |u_{0n} - u_0| \end{aligned}$$

Therefore, for any $T \ge 0$

$$\begin{aligned} \sup_{t\in[0,T]} |u_n(t) - u(t)| &\le |u_{0n} - u_0| \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus,

$$u_n(t) \to u(t) \text{ uniformly on } [0, T] \quad \forall\, T \ge 0. \tag{2.1.14}$$

By a similar argument, we have from (2.1.13) ,

$$\frac{\mathrm{d}u_n}{\mathrm{d}t}(t) \to \phi(t) \text{ uniformly on } [0,T] \quad \forall\, T \geq 0 \tag{2.1.15}$$

where $\phi(t)$ is the pointwise limit of $\frac{\mathrm{d}u_n}{\mathrm{d}t}(t)$. Therefore by Lemma 2.1.3, it follows that $u(t)$ is differentiable and $\phi(t) = \frac{\mathrm{d}u}{\mathrm{d}t}(t)$ on $[0,T]$ for any $T \geq 0$. Since u_n are continuous on $[0,T]$, so is the uniform limit u on $[0,T]$ $\forall\, T \geq 0$. That is, $u \in C([0,+\infty);H)$. Again, since $u_n \in C^1([0,+\infty);H)$, it follow that $\frac{\mathrm{d}u_n}{\mathrm{d}t}(t) \in C([0,+\infty);H)$, and thus, $\frac{\mathrm{d}u}{\mathrm{d}t}(t)$ being the uniform limit , it follows that $\frac{\mathrm{d}u}{\mathrm{d}t}(t)$ is continuous on $[0,T]$ $\forall\, T \geq 0$. That is, $\frac{\mathrm{d}u}{\mathrm{d}t}(t) \in C([0,+\infty);H)$. Hence,

$$u \in C^1([0,+\infty);H).$$

Now, we note that

$$Au_n(t) = -\frac{\mathrm{d}u_n}{\mathrm{d}t}(t) \to -\frac{\mathrm{d}u}{\mathrm{d}t}(t)$$

and

$$u_n(t) \to u(t).$$

But since A is a closed operator, it follows that $u(t) \in D(A)$ and $Au(t) = -\frac{\mathrm{d}u}{\mathrm{d}t}(t)$, that is,

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) + Au(t) = 0.$$

Moreover, note that $Au = -\frac{\mathrm{d}u}{\mathrm{d}t} \in C([0,+\infty);H)$. Also since $u \in C^1([0,+\infty);H)$, we have

$$\begin{aligned}
&|u(t) - u(t_0)|_{D(A)} \\
&= |u(t) - u(t_0)| + |Au(t) - Au(t_0)| \\
&\to 0 \text{ as } |t - t_0| \to 0 \text{ for any } t_0 \geq 0.
\end{aligned}$$

Thus,

$$u \in C([0,+\infty);D(A)).$$

This concludes the proof of the Theorem. ■

2.2 Regularity of the solutions

It turns out that if one were to make additional assumptions on the initial data u_0, the solution to the system (2.1.1) in Theorem 2.1.5 (Hille-Yosida Theorem) is more regular than just $C^1([0,+\infty);H)\bigcap C([0,+\infty);D(A))$. For that purpose we inductively define the space $D(A^k)=\{v\in D(A^{k-1}); Av\in D(A^{k-1})\}$, where $k\geq 2$ is an integer.

Claim : *$D(A^k)$ is a Hilbert space for the scalar product*

$$(u,v)_{D(A^k)}=\sum_{j=0}^{k}(A^j u, A^j v);$$

the corresponding norm is

$$|u|_{D(A^k)}=\left(\sum_{j=0}^{k}|A^j u|^2\right)^{\frac{1}{2}}.$$

Proof. Note that $(u,v)_{D(A^k)}$ being a finite sum of scalar products is indeed a scalar product itself. Let $(u_n)\subseteq D(A^k)$ be a Cauchy sequence. That is, given $\epsilon>0$, $\exists\ N_\epsilon$ such that $\forall\ m,n>N_\epsilon$,

$$\begin{aligned}
&|u_n-u_m|_{D(A^k)}<\epsilon\\
\Rightarrow&\left(\sum_{j=0}^{k}|A^j(u_n-u_m)|^2\right)^{\frac{1}{2}}<\epsilon\\
\Rightarrow&|u_n-u_m|^2+|Au_n-Au_m|^2+\cdots+|A^k u_n-A^k u_m|^2<\epsilon^2\quad\forall\ m,n>N_\epsilon\\
\Rightarrow&|u_n-u_m|^2<\epsilon^2\\
\Rightarrow&|u_n-u_m|<\epsilon
\end{aligned}$$

This means u_n is also Cauchy in H. Then u_n converges to some limit $u_0\in H$. Similarly, $(Au_n),(A^2u_n),\cdots,(A^k u_n)$ are Cauchy sequences in H and hence they converge to the limits, say $u_1,u_2,\cdots,u_k$ respectively.

Recall that A is maximal monotone and hence a closed operator. Note that $(u_n)\subset D(A^k)$ implies $(u_n)\subset D(A^j)$ for $j=0$ to k. Now $(u_n)\subset D(A)$, $u_n\to u_0\in H$, $Au_n\to u_1\in H$. Since A is closed, this means $u_0\in D(A)$ and $u_1=Au_0$.

Again, $(u_n)\subset D(A^2)$ and we just saw that $Au_n\to Au_0$. Also, $A(Au_n)=A^2u_n\to u_2$ Since A is closed, it follows that $Au_0\in D(A)$ and $u_2=A^2u_0$.

Repeating this argument inductively, it is clear that $u_j=A^j u_0$ and $A^j u_0\in D(A)$ for $j=0$ to k. This in turn implies that $u_0\in D(A^k)$. Thus the Cauchy sequence $(u_n)\to u_0$ in $D(A^k)$ and hence $D(A^k)$ is complete wrt the distance function induced by the scalar product. ∎

Theorem 2.2.1. *Suppose $u_0 \in D(A^k)$ for some integer $k \geq 2$. Then the solution u to the problem (2.1.1):*

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + Au = 0 \ \text{on} \ [0, +\infty), \\ u(0) = u_0, \end{cases}$$

satisfies

$$u \in C^{k-j}([0, +\infty); D(A^j)) \quad \forall\, j = 0, 1, \cdots, k.$$

Proof. Assume first $k = 2$. Denote by H_1 the Hilbert space $D(A)$ equipped with the scalar product $(u, v)_{D(A)} = (u, v) + (Au, Av)$ and the norm $|u|_{D(A)} = (|u|^2 + |Au|^2)^{1/2}$. Denote by A_1 the operator $A_1 : D(A_1) \subset H_1 \to H_1$ such that

$$D(A_1) = D(A^2),$$
$$A_1 v = Av \text{ for } v \in D(A_1) = D(A^2).$$

We will show that A_1 is maximal monotone in $H_1 = D(A)$.

Note that for $v \in D(A_1) = D(A^2) \subset D(A)$, $(A_1 v, v) = (Av, v) \geq 0$. Now, take any $f \in H_1 = D(A)$. Since A is maximal monotone, this means $\exists\, u \in D(A) = H_1$ such that $u + Au = f$. That is, $Au = f - u$. Now $f, u \in H_1 \Rightarrow Au = f - u \in H_1$.

But $u \in D(A)$ and $Au \in D(A)$ imply $u \in D(A^2)$. Thus for any $f \in H_1$, we have found $u \in D(A_1)$ such that $u + A_1 u = u + Au = f$. So A_1 is maximal monotone in H_1.

Consider the system

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + A_1 u = 0 \text{ on } [0, +\infty), \\ u(0) = u_0, \end{cases} \tag{2.2.1}$$

with maximal monotone operator A_1 in the space H_1. So we can apply Theorem 2.1.5 to the above system to obtain a solution $u \in C^1([0, +\infty); H_1) \bigcap C([0, +\infty); D(A_1))$. Note that $u(t) \in D(A_1) = D(A^2) \subset D(A)$. So $A_1 u(t) = Au(t)$. But then u satisfies

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + Au = 0 \text{ on } [0, +\infty), \\ u(0) = u_0. \end{cases}$$

By the uniqueness of the solution obtained in Theorem 2.1.5, this u is the unique solution to the system (2.1.1).

So we have that $u \in C^1([0,+\infty); D(A)) \bigcap C([0,+\infty); D(A^2))$. Now we will show that $u \in C^2([0,+\infty); H)$.

For any $x \in H_1$ with $|x|_{H_1} = 1$, we have

$$\begin{aligned} & |x|_{H_1} = (x,x)_{H_1} = 1 \\ \Rightarrow\ & |x|^2 + |Ax|^2 = 1 \\ \Rightarrow\ & |x|^2 = 1 - |Ax|^2 \leq 1. \end{aligned}$$

This is true for all $x \in H_1$ such that $|x|_{H_1} = 1$. So, $|Ax| \leq 1, \quad \forall\, x \in H_1$ such that $|x|_{H_1} = 1$ and therefore

$$\sup_{\substack{x \in H_1 \\ |x|_{H_1} = 1}} |Ax| \leq 1.$$

That is, when A is considered as an operator from H_1(as a Hilbert space with $(\cdot,\cdot)_{H_1}$), it is a a bounded linear operator(hence continuous). In other words,

$$A \in \mathcal{L}(H_1; H)$$

Again, we have from before that $u \in C([0,+\infty); H_1)$. Therefore $Au \in C([0,+\infty); H)$. We had previously shown that $u \in C([0,+\infty); D(A_1))$. Therefore,

$$\begin{aligned} & u \in C([0,+\infty); D(A_1)) \\ \Rightarrow\ & u(t) \in D(A_1) = D(A^2) \\ \Rightarrow\ & Au(t) \in D(A) \\ \Rightarrow\ & -\frac{\mathrm{d}u}{\mathrm{d}t}(t) \in D(A) = H_1. \end{aligned}$$

Now,

$$\begin{aligned}
&\lim_{h\to 0}\frac{Au(t+h)-Au(t)}{h}\\
=&\lim_{h\to 0}A\left(\frac{u(t+h)-u(t)}{h}\right)\\
=&A\left(\lim_{h\to 0}\frac{u(t+h)-u(t)}{h}\right) && \text{(since } A \text{ is linear bdd, hence continuous)}\\
=&A\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)\\
\therefore\ & \frac{\mathrm{d}}{\mathrm{d}t}(Au)=A\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right) && (2.2.2)\\
\therefore\ & \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)=\frac{\mathrm{d}}{\mathrm{d}t}(-Au)\\
=&-A\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)=-A\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)\in C([0,+\infty);H)\\
\Rightarrow\ &\frac{\mathrm{d}u}{\mathrm{d}t}\in C^1(0,+\infty];H)\\
\Rightarrow\ &u\in C^2([0,+\infty];H).
\end{aligned}$$

Also,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)=-A\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)\\
\Rightarrow\ &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)+A\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)=0 \quad \text{on } [0,+\infty). && (2.2.3)
\end{aligned}$$

Now we use induction for the general case $k \geq 3$. Assume the Theorem holds up to order $k-1$. Suppose $u_0 \in D(A^k)$. We have shown above that the solution u of the problem

$$\begin{cases}\dfrac{\mathrm{d}u}{\mathrm{d}t}+Au=0 \text{ on } [0,+\infty),\\ u(0)=u_0.\end{cases}$$

satisfies :

$$u \in C^2([0,+\infty);H)\bigcap C^1([0,+\infty);D(A)),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)+A\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)=0 \text{ on } [0,+\infty).$$

Let's denote $v=\frac{\mathrm{d}u}{\mathrm{d}t}$. Note that v satisfies the system:

$$\frac{\mathrm{d}v}{\mathrm{d}t}+Av=0 \text{ on } [0,+\infty),$$
$$v(0)=\frac{\mathrm{d}u}{\mathrm{d}t}(0)=-Au(0)=-Au_0. \tag{2.2.4}$$

By assumption, $u_0 \in D(A^k)$. So $-Au_0 \in D(A^{k-1})$. By Induction hypothesis the theorem holds for the system (2.2.4) with $v(0)=-Au_0 \in D(A^{k-1})$. Then

$$v \in C^{k-1-j}([0,+\infty);D(A^j)) \qquad \text{for } j=0,1,\cdots,k-1$$
$$\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}t} \in C^{k-1-j}([0,+\infty);D(A^j)) \qquad \text{for } j=0,1,\cdots,k-1$$
$$\Rightarrow u \in C^{k-j}([0,+\infty);D(A^j)) \qquad \text{for } j=0,1,\cdots,k-1.$$

Now we only need to verify that $u \in C([0,+\infty);D(A^k))$. For $j=k-1$, we have

$$u \in C^1([0,+\infty);D(A^{k-1})$$
$$\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}t} \in C([0,+\infty);D(A^{k-1}))$$
$$\Rightarrow Au \in C([0,+\infty);D(A^{k-1})).$$

So, we have, $u(t) \in D(A^{k-1})$ and $Au(t) \in D(A^{k-1})$. Therefore, $u(t) \in D(A^k)$ for all $t \geq 0$. Note that

$$|u(t)-u(t_0)|^2_{D(A^k)}=\sum_{j=0}^{k}|A^j(u(t)-u(t_0))|^2 \quad \text{for any } t_0 \geq 0.$$

And,

$$|Au(t)-Au(t_0)|_{D(A^{k-1})}=\sum_{j=0}^{k-1}|A^j(Au(t)-Au(t_0))|^2$$
$$=\sum_{j=1}^{k}|A^j(u(t)-u(t_0))|^2 \quad \text{for any } t_0 \geq 0.$$

Therefore,

$$|u(t)-u(t_0)|_{D(A^k)} = \left(|u(t)-u(t_0)|^2 + |Au(t)-Au(t_0)|^2_{D(A^{k-1})}\right)^{1/2}$$

$$\begin{aligned}\Rightarrow \lim_{t\to t_0}|u(t)-u(t_0)|_{D(A^k)} &= \left(\lim_{t\to t_0}|u(t)-u(t_0)|^2 + \lim_{t\to t_0}|Au(t)-Au(t_0)|^2\right)^{1/2}\\ &= 0\end{aligned}$$

(since $u \in C([0,+\infty);H)$ and $Au \in C([0,+\infty);D(A^{k-1}))$.)

Therefore, $u \in C([0,+\infty);D(A^k))$. This concludes the proof. ■

2.3 The case of self-adjoint operators

Suppose $A : D(A) \subseteq H \to H$ is an unbounded linear operator and $D(A)$ is dense in H, that is, $\overline{D(A)} = H$. Define $D(A^*)$ to be the set of all $f \in H$ such that the linear functional $g \mapsto (f, Ag)$ extends to a bounded linear functional on all of H. Since $D(A)$ is dense in H, by Riesz representation theorem it follows that there exists a unique $h \in H$ such that $(f, Ag) = (h, g)$. We define the **adjoint operator**, A^* of A as $A^*f = h$. Clearly, A^* is also a linear operator.

An operator A is called **symmetric** if $(u, Av) = (Au, v) \quad \forall\, u, v \in D(A)$. An operator A is called **self-adjoint** if $D(A) = D(A^*)$ and $A = A^*$.

Claim 1: *A is a symmetric operator if and only if*

$$\begin{aligned}&D(A) \subseteq D(A^*),\\ &A = A^* \text{ on } D(A).\end{aligned}$$

Proof. ($\Rightarrow$:)
A is symmetric. That is, $(u, Av) = (Au, v)$ for all $u, v \in D(A)$. Take $f \in D(A)$ and the functional $g \mapsto (f, Ag)$. Now, since A is symmetric, $(f, Ag) = (Af, g)$ for all $g \in D(A)$. Denote by T the functional:

$$T(g) = (Af, g) \quad \forall\, g \in H.$$

Note that for $g \in D(A)$, $T(g) = (Af, g) = (g, Af)$. And thus T is an extension of the functional $g \mapsto (f, Ag)$ on all of H. Also, note that $|(Af, g)| \le |Af||g|$ by Cauchy-Schwarz inequality. Hence, T is a bounded linear functional that is an extension of the functional $g \mapsto (f, Ag)$ on all of H. Therefore by definition of $D(A^*)$, $f \in D(A^*)$. The choice of $f \in D(A)$ was arbitrary. So, we have $D(A) \subseteq D(A^*)$.

Again, by Riesz's Representation Theorem, $\exists\ h \in H$ such that $T(g) = (h, g)$ for all $g \in H$. By definition, $A^*f = h$. But $T(g) = (Af, g) \quad \forall g \in H$. Therefore,

$$\begin{aligned} &(Af, g) = (A^*f, g) \quad \forall\ g \in H \\ \Rightarrow\ &Af = A^*f. \end{aligned}$$

This is true for any $f \in D(A)$. Therefore, $A = A^*$ on $D(A)$.

$(\Leftarrow :\ )$
We have

$$\begin{aligned} &D(A) \subseteq D(A^*), \\ &A = A^* \text{ on } D(A). \end{aligned}$$

Take any $u \in D(A)$. Then $u \in D(A^*)$. Then by definition of $D(A^*)$, the functional $g \mapsto (u, Ag)$ can be extended to a bounded linear functional T on all of H such that $T(g) = (A^*u, g) \quad \forall\ g \in H$.

Now $A = A^* \Rightarrow T(g) = (Au, g) \quad \forall\ g \in H$. So, for $g \in D(A)$,

$$\begin{aligned} &T(g) = (u, Ag) \qquad\qquad \text{(since } T \text{ is an extension of the functional)} \\ \Rightarrow\ &(Au, g) = (u, Ag). \end{aligned}$$

Therefore, A is symmetric. ■

Claim 2: *If A is a self-adjoint operator, then it is symmetric.*

Proof. A is self-adjoint. Therefore, $D(A^*) = D(A)$ and $A = A^*$. Take any $u \in D(A) = D(A^*)$. By definition of $D(A^*)$, the functional $g \mapsto (u, Ag)$ extends to bounded linear functional, say T on all of H. Then $T(g) = (A^*u, g) = (Au, g)$ for all $g \in H$. But since T is an extension , for all $g \in D(A)$, $(u, Ag) = T(g) = (Au, g)$. Thus for all $u, g \in D(A)$, $(u, Ag) = (Au, g)$. That is, A is symmetric. ■

Claim 3: *Suppose $T \in \mathcal{L}(H;H)$. Then T is symmetric if and only if it's self-adjoint.*

Proof. From Claim 2, we know that a self-adjoint operator is always symmetric. So we only need to prove $(\Rightarrow:)$
T is symmetric. Therefore, $(u, Tv) = (Tu, v)$ for all $u, v \in H$. Then from Claim 1, we have, $D(T) \subseteq D(T^*)$. That is, $H = D(T) = D(T^*) \subseteq H$. Therefore, $D(T) = D(T^*) = H$ and $T = T^*$ on $D(T) = H$. Therefore, T is self-adjoint. ■

Theorem 2.3.1. *Suppose A is a maximal monotone operator that is symmetric. Then A is self-adjoint.*

Proof. Denote $J_1 = (I + A)^{-1}$. Recall that $D(J_1) = H$, $R(J_1) = D(A)$, and $\|J_1\|_{\mathcal{L}(H)} \leq 1$. We will first show that J_1 is self-adjoint. Since J_1 is linear and bounded, by Claim 3, it suffices to show that J_1 is symmetric.

Take any $u, v \in H$. Denote $J_1 u = u_1, J_1 v = v_1$.Note that $u_1, v_1 \in D(A)$. Then $u = u_1 + Au_1, v = v_1 + Av_1$. So, $Au_1 = u - u_1, Av_1 = v - v_1$. A is symmetric. Therefore,

$$\begin{aligned} &(u_1, Av_1) = (Au_1, v_1) \\ \Rightarrow\ &(u_1, v - v_1) = (u - u_1, v_1) \\ \Rightarrow\ &(u_1, v) - (u_1, v_1) = (u, v_1) - (u_1, v_1) \\ \Rightarrow\ &(J_1 u, v) = (u, J_1 v) \end{aligned}$$

Thus J_1 is symmetric and hence, self-adjoint.

Take any $u \in D(A^*)$. Let $f = u + A^* u$, that is, $f - u = A^* u$. Recall that from the definition of adjoint operator, we have $(u, Ag) = (A^* u, g)$ for all $g \in D(A)$. Therefore, $(u, Ag) = (f - u, g) = (f, g) - (u, g)$, that is

$$(u, g + Ag) = (f, g) \quad \forall\, g \in D(A). \tag{2.3.1}$$

Recall that $D(J_1) = H, R(J_1) = D(A)$. Take any $w \in H$. Then $J_1 w \in D(A)$. Denote $v = J_1 w$. Then $v + Av = w$. Therefore, by (2.3.1),

$$\begin{aligned} &(u, v + Av) = (f, v) \\ \Rightarrow\ &(u, w) = (f, J_1 w) \quad \forall\, w \in H. \end{aligned}$$

Since we showed that J_1 is symmetric, we have

$$(f, J_1 w) = (J_1 f, w) = (u, w) \ \ \forall\, w \in H.$$

Taking $w = J_1 f - u$,

$$\begin{aligned} &(J_1 f, J_1 f - u) = (u, J_1 f - u) \\ \Rightarrow\ &(J_1 f - u, J_1 f - u) = 0 \\ \Rightarrow\ &J_1 f = u. \end{aligned}$$

Therefore, $u \in R(J_1) = D(A)$. Since the choice of $u \in D(A^*)$ was arbitrary, this means, $D(A^*) \subseteq D(A)$. Again, since A is symmetric, by Claim 1,

$$\begin{aligned} &D(A) \subseteq D(A^*), \\ &A = A^* \text{ on } D(A). \end{aligned}$$

Therefore, $D(A) = D(A^*)$ and $A = A^*$ on $D(A) = D(A^*)$. Hence, A is self-adjoint. ■

Theorem 2.3.2. *Suppose A is a maximal monotone operator that is self-adjoint. Then for any $u_0 \in H$, $\exists$ a unique solution*

$$u \in C([0, +\infty); H) \cap C^1((0, +\infty); H) \cap C((0, +\infty); D(A))$$

to the problem:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + Au = 0 \ \textit{on}\ (0, +\infty), \\ u(0) = u_0. \end{cases} \tag{2.3.2}$$

Furthermore, we have the following estimates:

$$\begin{aligned} &|u(t)| \le |u_0| \quad \forall\, t > 0, \\ &\left|\frac{\mathrm{d}u}{\mathrm{d}t}(t)\right| = |Au(t)| \le \frac{1}{t}|u_0| \quad \forall\, t > 0, \\ &u \in C^k((0, +\infty); D(A^l)) \quad \textit{for all non-negative integers } k, l. \end{aligned}$$

Proof. First we show uniqueness. Suppose u and $\bar{u}$ are two solutions to the system (2.3.2). We have

$$\begin{aligned}
&\frac{\mathrm{d}u}{\mathrm{d}t} - \frac{\mathrm{d}\bar{u}}{\mathrm{d}t} + Au - A\bar{u} = 0 \quad \text{ on } (0, +\infty) \\
\Rightarrow\ &\frac{\mathrm{d}(u-\bar{u})}{\mathrm{d}t} = -A(u-\bar{u}) \\
\Rightarrow\ &\left(\frac{\mathrm{d}(u-\bar{u})}{\mathrm{d}t}, u-\bar{u}\right) = -(A(u-\bar{u}), u-\bar{u}) \\
&\leq 0. \qquad\qquad \text{(since } A \text{ is monotone, } (Av, v) \geq 0, \quad \forall\, v \in D(A))
\end{aligned}$$

Denote $\phi(t) = u(t) - \bar{u}(t)$. Then similar to (2.1.⊛), we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|\phi(t)|^2 &= 2\left(\frac{\mathrm{d}\phi(t)}{\mathrm{d}t}, \phi(t)\right) \\
&= \left(\frac{\mathrm{d}(u-\bar{u})}{\mathrm{d}t}, u-\bar{u}\right) \\
&\leq 0 \quad \text{on } (0, +\infty).
\end{aligned}$$

Therefore, $|\phi|^2$ is non-increasing on $(0, +\infty)$. Since $u, \bar{u} \in C([0, +\infty); H)$, $\phi = u - \bar{u}$ is continuous on $[0, +\infty)$. Therefore,

$$\begin{aligned}
&|\phi(t+h)|^2 \leq |\phi(h)|^2 \text{ for } h \geq 0, \forall\, t \geq 0 \\
\Rightarrow\ &|\phi(t)|^2 \leq |\phi(0)|^2 = 0, \forall t \geq 0 \\
\Rightarrow\ &\phi \equiv 0 \text{ on } [0, +\infty).
\end{aligned}$$

Thus the solution to (2.3.2) is unique.

Now we prove the existence part. Let us first assume that $u_0 \in D(A^2)$ (that is $u_0 \in D(A), Au_0 \in D(A)$). Let u be the solution to the system:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} \text{ on } [0, +\infty), \\ u(0) = u_0, \end{cases}$$

obtained in Theorem 2.1.5. We will show that $\left|\frac{\mathrm{d}u}{\mathrm{d}t}\right| \leq \frac{1}{t}|u_0|\ \forall\ t > 0$. Recall that $J_\lambda = (I + \lambda A)^{-1}$ has $D(J_\lambda) = H$ and $A_\lambda = \frac{1}{\lambda}(I - J_\lambda)$ has $D(A_\lambda) = D(J_\lambda) = H$.

We showed in the proof of Theorem 2.3.1 that J_1 is symmetric. By a similar argument it follows that J_λ is symmetric and hence by Claim 3, that J_λ is self-adjoint, that is, $J_\lambda = J_\lambda^*$. Now,

$$\begin{aligned}(A_\lambda u, v) &= (\frac{1}{\lambda}(I - J_\lambda)u, v) \\ &= \frac{1}{\lambda}(u, v) - \frac{1}{\lambda}(J_\lambda u, v) \\ &= \frac{1}{\lambda}(u, v) - \frac{1}{\lambda}(u, J_\lambda v) && \text{(since } J_\lambda \text{ is symmetric)} \\ &= \left(u, \frac{1}{\lambda}(v - J_\lambda v)\right) \\ &= (u, A_\lambda v).\end{aligned}$$

Thus A_λ is symmetric and hence by Claim 3, it is self-adjoint as well, that is, $A_\lambda^* = A_\lambda$. Consider the following approximate problem that was used in the proof of Theorem 2.1.5:

$$\begin{cases}\dfrac{\mathrm{d}u_\lambda}{\mathrm{d}t} + A_\lambda u_\lambda = 0 \text{ on } [0, +\infty), \\ u(0) = u_0.\end{cases} \tag{2.3.3}$$

Recall that A_λ is Lipschitz, since $|A_\lambda v| \leq \frac{1}{\lambda}|v|, \ \forall \ v \in H, \forall \ \lambda > 0$. Then by Theorem 2.1.1(Cauchy-Lipschitz-Picard), the unique solution u_λ to (2.3.3) satisfies $u_\lambda \in C^1([0, +\infty); H)$. Now from (2.3.3),

$$\begin{aligned}&\frac{\mathrm{d}u_\lambda}{\mathrm{d}t} = -A_\lambda u_\lambda \\ \Rightarrow &\left(\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t), u_\lambda(t)\right) = -(A_\lambda u_\lambda(t), u_\lambda(t)).\end{aligned}$$

Again, similar to (2.1.⊛), we have,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_\lambda(t)|^2 = \left(\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t), u_\lambda(t)\right) = -(A_\lambda u_\lambda(t), u_\lambda(t)).$$

Integrating wrt to t over $[0, T]$, we get

$$\begin{aligned}&\frac{1}{2}(|u_\lambda(T)|^2 - |u_\lambda(0)|^2) = -\int_0^T (A_\lambda u_\lambda(t), u_\lambda(t))dt \\ \Rightarrow &\frac{1}{2}|u_\lambda(T)|^2 + \int_0^T (A_\lambda u_\lambda(t), u_\lambda(t))dt = \frac{|u_0|^2}{2}.\end{aligned} \tag{2.3.4}$$

Taking the scalar product of (2.3.3) with $t\frac{\mathrm{d}u}{\mathrm{d}t}$, we have

$$t\left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}\right|^2 + t\left(A_\lambda u_\lambda(t), \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)\right) = 0.$$

Integrating wrt to t over $[0,T]$, we get

$$\int_0^T \left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)\right|^2 tdt + \int_0^T \left(A_\lambda u_\lambda(t), \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)\right) tdt = 0. \tag{2.3.5}$$

Now,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(A_\lambda u_\lambda, u_\lambda\right)\\
&= \lim_{h\to 0} \frac{(A_\lambda u_\lambda(t+h), u_\lambda(t+h)) - (A_\lambda u_\lambda(t), u_\lambda(t))}{h}\\
&= \lim_{h\to 0} \frac{(A_\lambda u_\lambda(t+h) - A_\lambda u_\lambda(t), u_\lambda(t+h)) - (A_\lambda u_\lambda(t), u_\lambda(t+h) - u_\lambda(t))}{h}\\
&= \lim_{h\to 0} \left(A_\lambda\left(\frac{u_\lambda(t+h)-u_\lambda(t)}{h}\right), u_\lambda(t+h)\right) + \lim_{h\to 0}\left(A_\lambda u_\lambda(t), \frac{u_\lambda(t+h)-u_\lambda(t)}{h}\right)\\
&= \left(A_\lambda \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}, u_\lambda\right) + \left(A_\lambda u_\lambda, \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}\right)\\
&= \left(\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}, A_\lambda u_\lambda\right) + \left(A_\lambda u_\lambda, \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}\right) \qquad \text{(since } A_\lambda \text{ is self-adjoint)}\\
&= 2\left(A_\lambda u_\lambda, \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}\right).
\end{aligned} \tag{2.3.6}$$

From (2.3.6),

$$\begin{aligned}
&\int_0^T \left(A_\lambda u_\lambda, \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}\right) tdt\\
=&\frac{1}{2}\int_0^T \left[\frac{\mathrm{d}}{\mathrm{d}t}(A_\lambda u_\lambda, u_\lambda)\right] tdt\\
=&\frac{1}{2}\left[t\int \frac{\mathrm{d}}{\mathrm{d}t}(A_\lambda u_\lambda, u_\lambda)dt\right]_0^T - \frac{1}{2}\int_0^T \left(\int \frac{\mathrm{d}}{\mathrm{d}t}(A_\lambda u_\lambda, u_\lambda)dt\right)dt\\
&\qquad \text{(using integration by parts)}\\
=&\frac{T}{2}(A_\lambda u_\lambda(T), u_\lambda(T)) - \frac{1}{2}\int_0^T (A_\lambda u_\lambda, u_\lambda)dt.
\end{aligned} \tag{2.3.7}$$

By Lemma 2.1.1, $t \mapsto \left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)\right|$ is non-increasing. Therefore,

$$\int_0^T \left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)\right|^2 tdt \geq \left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(T)\right|^2 \int_0^T tdt = \left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(T)\right|^2 \frac{T^2}{2}. \tag{2.3.8}$$

From (2.3.7), we have

$$\begin{aligned}
&\int_0^T (A_\lambda u_\lambda, u_\lambda) dt \\
&= T(A_\lambda u_\lambda(T), u_\lambda(T)) - 2\int_0^T \left(A_\lambda u_\lambda, \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}\right) t dt \\
&= T(A_\lambda u_\lambda(T), u_\lambda(T)) + 2\int_0^T \left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}\right|^2 t dt \qquad \text{(using (2.3.5))} \\
&\geq T(A_\lambda u_\lambda(T), u_\lambda(T)) + T^2 \left|\frac{\mathrm{d}u}{\mathrm{d}t}(T)\right|^2. \qquad \text{(using (2.3.8))}
\end{aligned}$$

So,

$$\int_0^T (A_\lambda u_\lambda, u_\lambda) dt \geq T(A_\lambda u_\lambda(T), u_\lambda(T)) + T^2 \left|\frac{\mathrm{d}u}{\mathrm{d}t}(T)\right|^2. \qquad (2.3.9)$$

Now, from (2.3.4),

$$\begin{aligned}
\frac{|u_0|^2}{2} &= \frac{1}{2}|u_\lambda(T)|^2 + \int_0^T (A_\lambda u_\lambda, u_\lambda) dt \\
&\geq \frac{1}{2}|u_\lambda(T)|^2 + T(A_\lambda u_\lambda(T), u_\lambda(T)) + T^2 \left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(T)\right|^2. \qquad \text{(using (2.3.9))}
\end{aligned}$$

That is,

$$\begin{aligned}
&|u_\lambda(T)|^2 + 2T(A_\lambda u_\lambda(T), u_\lambda(T)) + 2T^2 \left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(T)\right|^2 \leq |u_0|^2 \\
\Rightarrow\ &|u_\lambda(T)|^2 + 2T(A_\lambda u_\lambda(T), u_\lambda(T)) + T^2 \left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(T)\right|^2 + T^2 \left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(T)\right|^2 \leq |u_0|^2 \\
\Rightarrow\ &|u_\lambda(T)|^2 + 2T(A_\lambda u_\lambda(T), u_\lambda(T)) + T^2|A_\lambda u_\lambda(T)|^2 + T^2 \left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(T)\right|^2 \leq |u_0|^2 \\
&\qquad \text{(using } \mathrm{d}u_\lambda/\mathrm{d}t = -A_\lambda u_\lambda) \\
\Rightarrow\ &\left|u_\lambda(T) + T\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(T)\right|^2 + T^2 \left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(T)\right|^2 \leq |u_0|^2 \\
\Rightarrow\ &T^2 \left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(T)\right|^2 \leq |u_0|^2 \\
\Rightarrow\ &\left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(T)\right| \leq \frac{1}{T}|u_0|, \quad \forall\, T > 0, \quad \forall\, \lambda > 0. \qquad (2.3.10)
\end{aligned}$$

Recall that in the proof of Theorem 2.1.5, we showed that

$$\frac{\mathrm{d}u_\lambda}{\mathrm{d}t} \to \frac{\mathrm{d}u}{\mathrm{d}t} \text{ uniformly as } \lambda \to 0^+.$$

So, passing on to the limit as $\lambda \to 0^+$, we get

$$\left|\frac{\mathrm{d}u}{\mathrm{d}t}(T)\right| \leq \frac{1}{T}|u_0| \quad \forall\, T > 0.$$

Recall from Lemma 2.1.4, we have $D(A^2)$ is dense in $D(A)$. But here we also have $D(A)$ is dense in H. So, $D(A^2)$ is dense in H. Therefore for any $u_0 \in H$, there exists a sequence $(u_{0n}) \subseteq D(A^2)$ such that $u_{0n} \to u_0$. Now consider the problem:

$$\begin{cases} \dfrac{\mathrm{d}w}{\mathrm{d}t} + Aw = 0 \text{ on } [0, +\infty), \\ w(0) = u_{0n}. \end{cases}$$

Thus by Theorem 2.1.5, the above problem has a solution, say u_n that satisfies $u_n \in C^1([0, +\infty); H) \bigcap C([0, +\infty); D(A))$ and furthermore , we have the estimate $|u_n(t)| \leq |u_0|$, $\forall\, t \geq 0$. Also from the preceding analysis, we have $\left|\frac{\mathrm{d}u_n}{\mathrm{d}t}(t)\right| \leq \frac{1}{t}|u_{0n}|$, $\forall\, t > 0$. Note that $u_n - u_m$ is a solution to the problem:

$$\begin{cases} \dfrac{\mathrm{d}w}{\mathrm{d}t} + Aw = 0, \\ w(0) = u_{0n} - u_{0m}. \end{cases}$$

And thus, we have the estimates:

$$\left|\frac{\mathrm{d}u_n}{\mathrm{d}t}(t) - \frac{\mathrm{d}u_m}{\mathrm{d}t}(t)\right| \leq \frac{1}{t}|u_{0n} - u_{0m}|, \; t > 0, \tag{2.3.11}$$

$$|u_n(t) - u_m(t)| \leq |u_{0n} - u_{0m}|, \; t \geq 0. \tag{2.3.12}$$

Thus $u_n \to u$ uniformly on $[0, T]$ for every $T \geq 0$. Note that $u_n \in C^1([0, +\infty); H)$, and hence, u being the uniform limit is also continuous on $[0, T]$, $\forall t \geq 0$. That is, $u \in C([0, +\infty); H)$. From (2.3.11) and Lemma 2.1.3, we see that u is differentiable on $[\delta, T + \delta]$ and $\frac{\mathrm{d}u_n}{\mathrm{d}t} \to \frac{\mathrm{d}u}{\mathrm{d}t}$ in $C([\delta, T + \delta]; H)$ for any $\delta > 0$ and $T > 0$. Hence

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) \in C((0, +\infty); H).$$

This along with $u \in C([0, +\infty); H)$ implies that $u \in C^1((0, +\infty); H)$. So,

$$u \in C([0, +\infty); H) \bigcap C^1((0, +\infty); H). \tag{2.3.14}$$

Again, note that

$$Au_n(t) = -\frac{\mathrm{d}u_n}{\mathrm{d}t}(t) \to -\frac{\mathrm{d}u}{\mathrm{d}t}(t)$$

and

$$u_n(t) \to u(t).$$

Since A is a closed operator (being maximal monotone), this means

$$u(t) \in D(A) \text{ and } Au(t) = -\frac{\mathrm{d}u}{\mathrm{d}t}(t),$$
$$\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}t}(t) + Au(t) = 0 \text{ on } (0, +\infty).$$

Now we will prove that $u \in C^k((0,+\infty); D(A^l))$ for all non-negative integers k, l. For that, we will first show that

$$u \in C^{k-j}((0,+\infty); D(A^j)) \quad \forall\, j = 0, 1, \cdots, k \text{ for all integers } k \geq 2. \qquad (2.3.15)$$

From (2.3.14), we have,

$$\begin{aligned}
&|u(t) - u(t_0)|_{D(A)} \\
&= (|u(t) - u(t_0)|^2 + |Au(t) - Au(t_0)|^2)^{1/2} \\
&\to 0 \text{ as } |t - t_0| \to 0, \quad \forall\, t_0 > 0.
\end{aligned}$$

This along with (2.3.14) shows that (2.3.15) holds for $k = 1$. We will proceed by induction on k. Now assume (2.3.15) holds for all non-negative integers upto $k - 1$.

Recall that we have already proved the first part of the Theorem , that is, given any $u_0 \in H$, the problem:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + Au = 0 \text{ on } (0, +\infty), \\ u(0) = u_0 \end{cases}$$

has a unique solution $u \in C([0,+\infty); H) \bigcap C^1((0,+\infty); H) \bigcap C((0,+\infty); D(A))$, where A is a maximal monotone and self-adjoint operator.

Now, let us denote, the Hilbert space $\tilde{H} = D(A^{k-1})$ with the norm we introduced in Section 2.3. Let's define the operator $\tilde{A}$ as follows:

$$\begin{aligned}
&\tilde{A} : D(\tilde{A}) \subseteq \tilde{H} \to \tilde{H} \text{ such that} \\
&D(\tilde{A}) = D(A^k), \\
&\tilde{A} = A \text{ on } D(\tilde{A}).
\end{aligned}$$

We claim that $\tilde{A}$ is maximal monotone in $\tilde{H}$. For all $v \in D(\tilde{A}) = D(A^k)$, we have

$$\begin{aligned}
&(\tilde{A}v, v)_{\tilde{H}} \\
&= (\tilde{A}v, v)_{D(A^{k-1})} \\
&= \sum_{j=0}^{k-1} (A^j(\tilde{A}v), A^j v) \\
&= \sum_{j=0}^{k-1} (A(A^j v), A^j v) && \text{(since } A = \tilde{A} \text{ on } D(\tilde{A})) \\
&\geq 0. && \text{(since } A \text{ is monotone)}
\end{aligned}$$

Now, take any $f \in \tilde{H} = D(A^{k-1})$. Then f is also in H. Since A is maximal monotone, we have $R(I + A) = H$. That is, $\exists\ v \in D(A)$ such that $v + Av = f$. $f \in D(A^{k-1}) \subseteq D(A)$ and $v \in D(A)$ imply $Av = f - v \in D(A)$. Therefore, $v \in D(A^2)$. Proceeding with this same line of argument inductively, it follows that $v \in D(A^{k-1})$, and together with $f \in D(A^{k-1})$, this means $Av = f - v \in D(A^{k-1})$, that is, $v \in D(A^k) = D(\tilde{A})$.

Thus for any $f \in \tilde{H}$, $\exists v \in D(\tilde{A})$, such that $(I + \tilde{A})v = f$, that is, $R(I + \tilde{A}) = \tilde{H}$. So, $\tilde{A}$ is maximal monotone.

Also, note that for all $u, v \in D(\tilde{A}) = D(A^k)$, $(\tilde{A}u, v) = (Au, v) = (u, Av) = (u, \tilde{A}v)$, since A is symmetric and $A = \tilde{A}$ on $D(\tilde{A})$. Thus $\tilde{A}$ is symmetric.

So by Theorem 2.3.1, $\tilde{A}$ being maximal monotone and symmetric, is also self-adjoint. Then by the first part of the Theorem, it follows that for any $v_0 \in \tilde{H}$, the system:

$$\begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}t} + \tilde{A}v = 0, \\ v(0) = v_0 \end{cases} \tag{2.3.16}$$

has a unique solution $v \in C([0, +\infty); \tilde{H}) \bigcap C^1((0, +\infty); \tilde{H}) \bigcap C((0, +\infty); D(\tilde{A}))$.

Note that by the induction hypothesis, the solution u of the first part of the theorem satisfies $u(\epsilon) \in D(A^{k-1})\ \forall \epsilon > 0$. Denote $u_\epsilon(t) = u(t + \epsilon)$ for $t > 0$. Then putting $v(0) = u(\epsilon)$, we see that $u_\epsilon(t)$ is the unique solution to the system (2.3.16) , and hence $u_\epsilon \in C([0, +\infty); \tilde{H}) \bigcap C^1((0, +\infty); \tilde{H}) \bigcap C((0, +\infty); D(\tilde{A}))$.

In particular,

$$\begin{aligned}
&u_\epsilon \in C((0,+\infty); D(\tilde{A})) = C((0,+\infty); D(A^k)) \quad \forall\, \epsilon > 0 \\
\Rightarrow &u \in C((\epsilon,+\infty); D(A^k)) \quad \forall\, \epsilon > 0 \\
\Rightarrow &u \in C((0,+\infty); D(A^k)).
\end{aligned}$$

Therefore $u_0 \in D(A^k)$ and u satisfies

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + Au = 0, \\ u(0) = u_0. \end{cases}$$

Then by Theorem 2.2.1, it follows that

$$u \in C^{k-j}((0,+\infty); D(A^j)) \quad \forall\, j = 0, 1, \cdots, k. \tag{2.3.17}$$

Thus by induction, (2.3.17) holds for all integers $k \geq 2$. And hence,

$$u \in C^k((0,+\infty); D(A^l))$$

for all non-negative integers k, l. This concludes the proof of the Theorem. ■

Chapter 3

Applications of Hille-Yosida Theorem

So far we have presented the necessary tools in Chapter 1, and our central theorem, the Hille-Yosida theorem and its related results in Chapter 2. Armed with all that, we are now ready to investigate their applications to some real-world phenomena. We will investigate three applications, namely, the Heat equation, the Wave equation and the problem of coupled sound and heat flow in compressible fluids.

3.1 Heat Equation

In this section, we will investigate the Heat equation that describes the distribution of heat (or variation in temperature) over time in a fixed region of space. First we will establish some notations. Here we take $\Omega \subseteq \mathbb{R}^N$, an open set with boundary $\Gamma = \partial\Omega$. We are concerned with the problem of finding a function $u(x,t) : \overline{\Omega} \times [0, \infty) \to \mathbb{R}$ that satisfies:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = 0 & \text{on } \ \Omega \times (0, \infty) & (3.1.1a) \\ u = 0 & \text{on } \ \Gamma \times (0, \infty) & (3.1.1b) \\ u(x,0) = u_0(x) & \text{on } \ \Omega & (3.1.1c) \end{cases}$$

where t is the time variable, and $\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$ denotes the Laplacian in the space variables x_i.

Here $u(x,t)$ is the temperature as a function of the spatial variables x_i and time t, $\frac{\partial u}{\partial t}$ denotes the rate of change of temperature at a point over time. The heat equation follows from Fourier's law of conduction. The heat equation and its variations also occur in other diffusion phenomena.

(3.1.1b) is the Dirichlet boundary condition which means that the boundary, Γ is at zero temperature.

In order to apply the Hille-Yosida theorem, we will consider $u(t)$ as the function $x \mapsto u(x,t)$ that belongs to a space of functions depending only on x, say for example, a function space $H = L^2(\Omega)$ or $H = H_0^1(\Omega)$. Also, for simplicity, we will assume Ω is of class C^∞, and $\Gamma = \partial\Omega$ is bounded. Now we will state and prove some results regarding the existence, uniqueness and regularity of solutions to (3.1.1a - c). We will follow the same argument presented in [2].

Theorem 3.1.1. *Suppose $u_0 \in L^2(\Omega)$. Then the problem (3.1.1a-c) has a unique solution, $u(x,t)$ satisfying*

$$u \in C([0,\infty); L^2(\Omega)) \cap C((0,\infty); H^2(\Omega) \cap H_0^1(\Omega)), \tag{3.1.2}$$

$$u \in C^1((0,\infty); L^2(\Omega)). \tag{3.1.3}$$

Furthermore,

$$u \in C^\infty(\overline{\Omega} \times [\epsilon, \infty)), \ \forall \epsilon > 0. \tag{3.1.4}$$

Also, $u \in L^2(0,\infty; H_0^1(\Omega))$ and

$$\frac{1}{2}\left\|u(T)\right\|_{L^2}^2 + \int_0^T \left\|\nabla u(t)\right\|_{L^2}^2 \mathrm{d}t = \frac{1}{2}\left\|u_0\right\|_{L^2}^2 \quad \forall\, T > 0. \tag{3.1.5}$$

Proof. Consider the unbounded linear operator $A : D(A) \subset H \to H$, where

$$\begin{cases} H = L^2(\Omega), \\ D(A) = H^2(\Omega) \cap H_0^1(\Omega), \\ Au = -\Delta u. \end{cases}$$

We know by Corollary 1.7.3 that A is a maximal monotone operator. Now for $u, v \in D(A)$, we have by Green's identity (1.6.2) ,

$$(Au, v) = \int_\Omega (-\Delta u)v = \int_\Omega \nabla u \cdot \nabla v,$$
$$(u, Av) = \int_\Omega u(-\Delta v) = \int_\Omega \nabla u \cdot \nabla v.$$

This shows that A is symmetric as well. Then by Theorem 2.3.1, we have that A is self-adjoint. So now, we can use Theorem 2.3.2 to conclude that the solution u to

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + Au = 0 \quad \text{on } (0,\infty), \\ u(0) = u_0 \in H \end{cases}$$

satisfies

$$u \in C([0,\infty);H) \cap C^1((0,\infty);H) \cap C((0,\infty);D(A)), \tag{3.1.7}$$

$$u \in C^k((0,\infty);D(A^l)) \ \ \forall\, k,l \in \mathbb{N}. \tag{3.1.8}$$

Note that (3.1.5) is equivalent to (3.1.1a-c). The condition $u = 0$ on Γ has been incorporated into the definition of $D(A)$ since $u \in D(A) \subset H_0^1(\Omega) \Rightarrow u|_\Gamma = 0$. Also note that (3.1.2) and (3.1.3) follow from (3.1.7).

Note that $D(A) = H^2(\Omega) \cap H_0^1(\Omega) = \{u \in H^2(\Omega) \mid u = 0 \text{ on } \Gamma\}$. Suppose

$$D(A^{l-1}) = \{u \in H^{2l-2} | u = \Delta u = \cdots = \Delta^{l-2} u = 0 \text{ on } \Gamma\}. \tag{3.1.9}$$

Recall that $D(A^l) = \{u \in D(A^{l-1}) | Au \in D(A^{l-1})\}$. Now, $u \in D(A^{l-1})$

$$\Rightarrow u \in H^{2l-2}(\Omega); u = \Delta u = \cdots = \Delta^{l-2} u = 0 \text{ on } \Gamma \tag{3.1.10}$$

and $Au \in D(A^{l-1})$

$$\Rightarrow \Delta u \in H^{2l-2}(\Omega); \Delta u = \Delta^2 u = \cdots = \Delta^{l-1} u = 0 \text{ on } \Gamma. \tag{3.1.11}$$

By Remark 1.7.2, there exists a unique f_u such that $-\Delta u + u = f_u \in H^{2l-2}(\Omega)$ and $u \in H^{2l}(\Omega)$ (since Ω is of class C^m for all $m \in \mathbb{N}$). And then, from (3.1.10) and (3.1.11), we can conclude that

$$D(A^l) \subset \{u \in H^{2l}(\Omega) | u = \Delta u = \cdots = \Delta^{l-1} u = 0 \text{ on } \Gamma\}. \tag{3.1.12}$$

On the other hand,

$$\begin{gathered} u \in H^{2l}(\Omega); u = \Delta u = \cdots = \Delta^{l-1} u = 0 \text{ on } \Gamma \\ \Rightarrow Au = -\Delta u \in H^{2l-2}; Au = \cdots = \Delta^{l-2}(Au) = 0 \text{ on } \Gamma. \end{gathered}$$

Thus by our induction hypothesis (3.1.9) we can conclude that $u \in D(A^{l-1}), Au \in D(A^{l-1})$, that is, $\{u \in H^{2l}(\Omega) | u = \Delta u = \cdots = \Delta^{l-1} u = 0 \text{ on } \Gamma\} \subseteq D(A^l)$. And together with (3.1.12), this means $D(A^l) = \{u \in H^{2l}(\Omega) | u = \Delta u = \cdots = \Delta^{l-1} u = 0 \text{ on } \Gamma\}$ for all integers $l \geq 1$.

Now we will show that $D(A^l) \subset H^{2l}$ is a continuous injection for all integers $l \geq 1$. It suffices to show that $\|u\|_{H^{2l}} \to 0$ as $\|u\|_{D(A^l)} \to 0$ for all integers $l \geq 1$.

Recall that from Remark 1.7.2, we have that given $u \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists a unique $f_u \in L^2(\Omega)$ such that $-\Delta u + u = f_u$. Moreover,

$$\|u\|_{H^2}^2 \leq 2C^2 \left(\|\Delta u\|_{L^2}^2 + \|u\|_{L^2}^2 \right)$$

where C is a constant that depends only on Ω. For $l = 1$, we have

$$\begin{aligned} & \|u\|_{D(A)} \to 0 \\ \Rightarrow \ & \|u\|_{L^2}, \|\Delta u\|_{L^2} \to 0 \\ \Rightarrow \ & \|u\|_{H^2} \to 0. \end{aligned}$$

Thus $D(A) \subset H^2(\Omega)$ is a continuous injection. Suppose that the statement holds for $l-1$, that is, $D(A^{l-1}) \subset H^{2l-2}(\Omega)$ is a continuous injection. Now, take $u \in D(A^l) \subset H^{2l}(\Omega) \subset H^{2l-2}(\Omega)$. So $-\Delta u \in H^{2l-2}(\Omega)$. And in that case $f_u = -\Delta u + u \in H^{2l-2}(\Omega)$. Hence from Theorem 1.7.2 and Remark 1.7.2, we can conclude that

$$\|u\|_{H^{2l}}^2 \leq 2C^2 \left(\|\Delta u\|_{H^{2l-2}}^2 + \|u\|_{H^{2l-2}}^2 \right). \tag{3.1.13}$$

Note that

$$\begin{aligned} \|u\|_{D(A^l)}^2 &= \sum_{j=0}^{l} \left\| A^j u \right\|_{L^2}^2 \\ &= \|u\|_{L^2}^2 + \|Au\|_{D(A^{l-1})}^2 \\ &= \|u\|_{D(A^{l-1})}^2 + \|Au\|_{L^2}^2. \end{aligned}$$

Thus

$$\begin{aligned} & \|u\|_{D(A^l)} \to 0 \\ \Rightarrow \ & \|u\|_{H^{2l-2}}, \|\Delta u\|_{H^{2l-2}} \to 0 && \text{(from induction hypothesis)} \\ \Rightarrow \ & \|u\|_{H^{2l}} \to 0. && \text{(from 3.1.13)} \end{aligned}$$

Thus we have shown $D(A^l) \subset H^{2l}(\Omega)$ is a continuous injection for all integers $l \geq 1$. Then by (3.1.7), we have

$$u \in C^k((0,\infty); H^{2l}(\Omega)) \quad \forall\, k, l \in \mathbb{N}. \tag{3.1.14}$$

It follows from Corollary 1.5.3, that given $k \in \mathbb{N}$, there exists $l > k$ such that $H^l(\Omega) \subset C^k(\overline{\Omega})$ is a continuous injection. Then clearly, $H^{2l}(\Omega) \subset H^l(\Omega) \subset C^k(\overline{\Omega})$ is a continuous injection and from (3.1.14), we have

$$u \in C^k((0,\infty); C^k(\overline{\Omega})) \quad \forall\, k \in \mathbb{N}.$$

Hence we can conclude that

$$u \in C^\infty(\overline{\Omega} \times [\epsilon, \infty)) \quad \forall\, \epsilon > 0.$$

Now, consider the function $\varphi(t) = \frac{1}{2}\|u(t)\|_{L^2}^2$. We have already shown that $u \in C^1((0,\infty); L^2(\Omega))$ and thus φ is C^1 on $(0,\infty)$. Thus $\forall\, t > 0$,

$$\begin{aligned}
\varphi'(t) &= \left(u(t), \frac{\mathrm{d}u}{\mathrm{d}t}(t)\right)_{L^2} \\
&= (u(t), \Delta u(t))_{L^2} \\
&= \int_\Omega u(t)\Delta u(t) \\
&= -\int_\Omega \nabla u(t)\cdot\nabla u(t) \qquad \text{(by Green's identity (1.6.2))} \\
&= -\|\nabla u(t)\|_{L^2}^2 .
\end{aligned}$$

Now integrating $\varphi'(t)$ on $[\epsilon, T]$ for $0 < \epsilon < T < \infty$, we have

$$\begin{aligned}
\varphi(T) - \varphi(\epsilon) &= \int_\epsilon^T \varphi'(t)\mathrm{d}t \\
&= -\int_\epsilon^T \|\nabla u(t)\|_{L^2}^2 \,\mathrm{d}t. \qquad (3.1.15)
\end{aligned}$$

Now since $u \in C([0,\infty); L^2(\Omega))$, $\varphi(\epsilon) \to \varphi(0) = \frac{1}{2}\|u_0\|_{L^2}^2$ as $\epsilon \to 0$. Then from (3.1.15), we have

$$\varphi(T) + \int_0^T \|\nabla u(t)\|_{L^2}^2 \,\mathrm{d}t = \frac{1}{2}\|u_0\|_{L^2}^2 \quad \forall\, T > 0.$$

That is, (3.1.5) holds.

Recall that the norm on $H^1(\Omega)$ is:

$$\|u(t)\|_{H^1}^2 = \|\nabla u(t)\|_{L^2}^2 + \|u(t))\|_{L^2}^2.$$

Therefore, for all $T > 0$,

$$\begin{aligned}
\int_0^T \|u(t)\|_{H^1}^2 &= \int_0^T \|\nabla u(t)\|_{L^2}^2 \,dt + \int_0^T \|u(t)\|_{L^2}^2 \,dt \\
&= \int_0^T \|\nabla u(t)\|_{L^2}^2 \,dt + 2\int_0^T \varphi(t)dt \\
&\le \frac{1}{2}\|u_0\|_{L^2}^2 + 2\int_0^T \varphi(t)dt. \qquad \text{(from (3.1.5))}
\end{aligned}$$

And thus

$$u \in L^2(0,\infty; H_0^1(\Omega)). \qquad (3.1.16)$$

This concludes the proof of Theorem 3.1.1. ■

The solution u of (3.1.1a - c) becomes more regular if we make additional assumptions on the intital data u_0. We deal with this in the next theorem.

Theorem 3.1.2. *(i) Suppose $u_0 \in H_0^1(\Omega)$, then the solution u of (3.1.1a - c) satisfies*

$$u \in C([0,\infty); H_0^1(\Omega)) \cap L^2(0,\infty; H^2(\Omega))$$

and

$$\frac{\partial u}{\partial t} \in L^2(0,\infty; L^2(\Omega)).$$

Furthermore, we have

$$\int_0^T \left\| \frac{\partial u}{\partial t}(t) \right\|_{L^2}^2 dt + \frac{1}{2} \|\nabla u(T)\|_{L^2}^2 = \frac{1}{2} \|\nabla u_0\|_{L^2}^2 . \tag{3.1.17}$$

(ii) Suppose $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, then the solution u satisfies

$$u \in C([0,\infty); H^2(\Omega)) \cap L^2(0,\infty; H^3(\Omega))$$

and

$$\frac{\partial u}{\partial t} \in L^2(0,\infty; H_0^1(\Omega)).$$

(iii) Suppose $u_0 \in H^k(\Omega) \quad \forall k \in \mathbb{N}$ and satisfies the compatibility conditions

$$\Delta^j u_0 = 0 \quad on\ \Gamma \quad \forall j \in \mathbb{N} \tag{3.1.18}$$

then $u \in C^\infty(\overline{\Omega} \times [0,\infty))$.

Proof of (i). Consider the space $H_1 = H_0^1(\Omega)$ equipped with the scalar product

$$(u,v)_{H_1} = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega uv.$$

Consider the unbounded linear operator $A_1 : D(A_1) \subset H_1 \to H_1$ defined by

$$\begin{cases} D(A_1) = \{u \in H^3(\Omega) \cap H_0^1(\Omega) | \Delta u \in H_0^1(\Omega)\}, \\ A_1 u = -\Delta u. \end{cases}$$

Now for $u \in D(A_1)$,

$$\begin{aligned} (Au,u)_{H_1} &= \int_\Omega \nabla(-\Delta u) \cdot \nabla u + \int_\Omega (-\Delta u) u \\ &= \int_\Omega |\Delta u|^2 + \int_\Omega (-\Delta u) u && \text{(using Green's identity (1.6.2))} \\ &= \int_\Omega |\Delta u|^2 + \int_\Omega |\nabla u|^2 && \text{(using Green's identity (1.6.2))} \\ &\geq 0. \end{aligned}$$

Thus A_1 is monotone. Now take any $f \in H_0^1(\Omega)$. By Theorem 1.7.2 and Remark 1.7.1, we have that there exists a unique $u \in H^3(\Omega) \cap H_0^1(\Omega)$ such that $-\Delta u + u = f$. Again since $f \in H_0^1(\Omega)$, this means $\Delta u \in H_0^1(\Omega)$, this means $\Delta u \in H_0^1(\Omega)$ and so, $u \in D(A_1)$. Thus, $R(A_1 + I) = H_1$, that is A_1 is maximal montone. Now for all $u, v \in D(A_1)$, we have by repeated use of Green's identity (1.6.2):

$$\begin{aligned}
(A_1 u, v)_{H_1} &= \int_\Omega \nabla(-\Delta u) \cdot \nabla v + \int_\Omega (-\Delta u) v \\
&= \int_\Omega (\Delta u)(\Delta v) + \int_\Omega \nabla u \cdot \nabla v \\
&= \int_\Omega \nabla u \cdot \nabla(-\Delta v) + \int_\Omega u(-\Delta v) \\
&= (u, A_1 v)_{H_1}.
\end{aligned}$$

Thus A_1 is symmetric. Then by Theorem 2.3.1, we have that A_1 is self-adjoint. So now we can use Theorem 2.3.2 to conclude that for $u_0 \in H_0^1(\Omega)$, there exists a unique solution u to

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + A_1 u = 0, \\ u(0) = u_0, \end{cases}$$

which satisfies

$$u \in C([0,\infty); H_1) \cap C^1((0,\infty); H_1) \cap C((0,\infty); D(A_1)), \tag{3.1.19}$$

$$u \in C^k((0,\infty); D(A^l)) \quad \forall\, k, l \in \mathbb{N}. \tag{3.1.20}$$

In particular, we have

$$\begin{aligned}
&u \in C^k((0,\infty); H_0^1(\Omega)) \ \ \forall\, k \in \mathbb{N} \\
\Rightarrow\ &u \text{ is } C^\infty \text{ on } (0,\infty).
\end{aligned}$$

Let $\varphi(t) = \frac{1}{2}\|\nabla u(t)\|_{L^2}^2$. Note that φ is C^∞ on $(0,\infty)$. Now,

$$\begin{aligned}
\varphi'(t) &= \left(\nabla u(t), \frac{\mathrm{d}}{\mathrm{d}t}(\nabla u(t))\right)_{L^2} \\
&= \int_\Omega \left(\nabla u(t) \cdot \nabla(\frac{\mathrm{d}u}{\mathrm{d}t}(t))\right) \\
&= -\int_\Omega (\Delta u(t))(\frac{\mathrm{d}u}{\mathrm{d}t}(t)) && \text{(by Green's identity (1.6.2))} \\
&= -\int_\Omega \left(\frac{\mathrm{d}u}{\mathrm{d}t}(t)\right)\left(\frac{\mathrm{d}u}{\mathrm{d}t}(t)\right) && \text{(since } u \text{ is a solution of } \tfrac{\mathrm{d}u}{\mathrm{d}t} - \Delta u = 0) \\
&= -\left\|\frac{\mathrm{d}u}{\mathrm{d}t}(t)\right\|_{L^2}^2. && (3.1.21)
\end{aligned}$$

For $0 < \epsilon < T < \infty$, integrating on $[\epsilon, T]$, we have from (3.1.21):

$$\varphi(T) - \varphi(\epsilon) + \int_\epsilon^T \left\| \frac{\mathrm{d}u}{\mathrm{d}t}(t) \right\|_{L^2}^2 = 0. \tag{3.1.22}$$

Now, from (3.1.19), we have $u \in C([0, \infty); H_1)$. Therefore, $\|u(t) - u(t_0)\|_{H_1} \to 0$ as $|t - t_0| \to 0$ for any $t_0 \in [0, \infty)$. Recall that

$$\|u(t) - u(t_0)\|_{H_1}^2 = \|\nabla(u(t) - u(t_0))\|_{L^2}^2 + \|u(t) - u(t_0)\|_{L^2}^2 .$$

Therefore,

$$\begin{aligned} &\|u(t) - u(t_0)\|_{H_1} \to 0 \\ \Rightarrow\ &\|\nabla(u(t) - u(t_0))\|_{L^2}^2 \to 0 \end{aligned}$$

for any $t_0 \in [0, \infty)$. Thus $\nabla u \in C([0, \infty); L^2(\Omega))$. Hence we have that $\varphi(\epsilon) \to \varphi(0) = \frac{1}{2}\|\nabla u_0\|_{L^2}^2$ as $\epsilon \to 0$. Then taking the limit as $\epsilon \to 0$, we have from (3.1.22) that

$$\varphi(T) - \varphi(0) + \int_0^T \left\| \frac{\mathrm{d}u}{\mathrm{d}t}(t) \right\|_{L^2}^2 dt = 0. \tag{3.1.23}$$

That is, (3.1.7) holds. Now note that from (3.1.23), we have for all $T > 0$,

$$\begin{aligned} \int_0^T \|\Delta u(t)\|_{L^2}^2 \, dt &= \int_0^T \left\| \frac{\mathrm{d}u}{\mathrm{d}t}(t) \right\|_{L^2}^2 dt \\ &= \varphi(0) - \varphi(T) \\ &\le \varphi(0). \qquad\qquad (\text{since } \varphi(T) \ge 0, \forall T > 0) \end{aligned}$$

That is, $\Delta u \in L^2(0, \infty; L^2(\Omega))$. Recall that from Theorem 3.1.1, we already have $u \in L^2(0, \infty; H_0^1(\Omega))$ and that implies $u \in L^2(0, \infty; L^2(\Omega))$.

Now, recall that by Remark 1.7.2,

$$\|u\|_{H^2}^2 \le 2C^2 \left(\|\Delta u\|_{L^2}^2 + \|u\|_{L^2}^2 \right)$$

where C is a constant that depends only on Ω. Therefore, for any $T > 0$,

$$\begin{aligned} \int_0^T \|u\|_{H_2}^2 \, dt &\le 2C^2 \left(\int_0^T \|\Delta u\|_{L^2}^2 \, dt + \int_0^T \|u\|_{L^2}^2 \, dt \right) \\ &< \infty. \end{aligned}$$

The last inequality holds since $u \in L^2(0, \infty; L^2(\Omega))$, $\Delta u \in L^2(0, \infty; L^2(\Omega))$. And hence, $u \in L^2(0, \infty; H^2(\Omega))$. This concludes the proof of (i).

■

Proof of (ii). Here, we consider the space $H_2 = H^2(\Omega) \cap H_0^1(\Omega)$ equipped with the scalar product

$$(u, v)_{H_2} = (\Delta u, \Delta v)_{L^2} + (u, v)_{L^2}.$$

The corresponding norm is equivalent to the usual H^2 norm. To see this note that

$$\|u\|_{H^2}^2 = \|u\|_{H^1}^2 + \sum_{|\alpha|=2} \|D^\alpha u\|_{L^2}^2 \geq \|u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 = \|u\|_{H_2}^2$$

and

$$\begin{aligned} \|u\|_{H^2}^2 &\leq 2C^2 \left(\|u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2\right) && \text{(by Remark 1.7.2)} \\ &= 2C^2 \|u\|_{H_2}^2 . \end{aligned}$$

Consider the unbounded linear operator $A_2 : D(A_2) \subset H_2 \to H_2$ defined by

$$\begin{cases} D(A_2) = \{u \in H^4(\Omega) | u \in H_0^1(\Omega), \Delta u \in H_0^1(\Omega)\}, \\ A_2 u = -\Delta u. \end{cases}$$

<u>*Claim 3.1.1:*</u> *Let $A : D(A) \subset H \to H$ be a self-adjoint maximal monotone operator on the Hilbert space H. Consider the Hilbert space $\tilde{H} = D(A)$ with the scalar product $(u, v)_{\tilde{H}} = (u, v)_H + (Au, Av)_H$. Then the operator $\tilde{A} : D(\tilde{A}) \subset \tilde{H} \to \tilde{H}$ defined by*

$$\begin{cases} D(\tilde{A}) = D(A^2) \\ \tilde{A}u = Au \end{cases}$$

is a self-adjoint maximal monotone operator.

Proof of Claim 3.1.1. $u \in D(\tilde{A}) = D(A^2) \Rightarrow u \in D(A), Au \in D(A)$. Therefore,

$$\begin{aligned} (\tilde{A}u, u)_{\tilde{H}} &= (Au, u)_{\tilde{H}} \\ &= (Au, u)_H + (A(Au), Au)_H \\ &\geq 0. && \text{(since A is monotone)} \end{aligned}$$

That is, $\tilde{A}$ is monotone. Now, take $f \in \tilde{H} = D(A) \subset H$. Then $\exists u \in D(A)$ such that $u + Au = f$ since A is maximal monotone. Then $Au = f - u \in D(A)$, and hence $u \in D(A^2) = D(\tilde{A})$. Therefore, $R(I + \tilde{A}) = \tilde{H}$, and thus $\tilde{A}$ is maximal monotone. Take $u, v \in D(\tilde{A}) = D(A^2)$. Then,

$$\begin{aligned} (\tilde{A}u, v)_{\tilde{H}} &= (Au, v)_H + (A(Au), Av)_H \\ &= (u, Av)_H + (Au, A(Av))_H && \text{(since A is symmetric)} \\ &= (u, \tilde{A}v)_{\tilde{H}}. \end{aligned}$$

Thus, $\tilde{A}$ is symmetric and being maximal monotone, it is also self-adjoint by Theorem 2.3.1. ■

Continuing with the proof of (ii), recall that the operator $A : D(A) \subset H \to H$ defined by

$$\begin{cases} H = L^2(\Omega), \\ D(A) = H^2(\Omega) \cap H_0^1(\Omega), \\ Au = -\Delta u, \end{cases}$$

is self-adjoint maximal-monotone as shown in the proof of Theorem 3.1.1. Now, note that here

$$\begin{aligned} &H_2 = D(A), \\ &D(A_2) &&= \{u \in H^4(\Omega) | u, \Delta u \in H_0^1(\Omega)\} \\ & &&= \{u \in H^2(\Omega) \cap H_0^1(\Omega) | \Delta u \in H^2(\Omega) \cap H_0^1(\Omega)\} \\ & &&= \{u \in D(A) | Au \in D(A)\} \\ & &&= D(A^2). \end{aligned}$$

Also, $(u, v)_{H_2} = (u, v)_{L^2} + (\Delta u, \Delta v)_{L^2} = (u, v)_{L^2} + (Au, Av)_{L^2}$. Therefore, we can apply Claim 3.1.1 to conclude that A_2 is self-adjoint maximal monotone. Then it follows from Theorem 2.3.2 that the solution u to

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + A_2 u = 0 \text{ on } (0, \infty) \\ u(0) = u_0 \in H_2 \end{cases}$$

satisfies

$$u \in C([0, \infty); H_2) \cap C^1((0, \infty); H_2) \cap C((0, \infty); D(A_2)), \tag{3.1.24}$$

$$u \in C^k((0, \infty); D(A^l)) \quad \forall\, k, l \in \mathbb{N}. \tag{3.1.25}$$

In particular $u \in C^k((0, \infty); H^2(\Omega) \cap H_0^1(\Omega))$ for all $k \in \mathbb{N}$. That is, u is C^∞ on $(0, \infty)$. Consider $\varphi(t) = \frac{1}{2} \|\Delta u(t)\|_{L^2}^2$. Then φ is C^∞ on $(0, \infty)$. We have

$$\begin{aligned} \varphi'(t) &= \left(\Delta u(t), \frac{\mathrm{d}}{\mathrm{d}t}(\Delta u(t))\right)_{L^2} \\ &= \left(\Delta u(t), \Delta(\frac{\mathrm{d}u}{\mathrm{d}t}(t))\right)_{L^2} \\ &= \left(\Delta u(t), \Delta^2 u(t)\right)_{L^2} && \text{(since } \tfrac{\mathrm{d}u}{\mathrm{d}t} - \Delta u = 0) \\ &= -\int_\Omega (\nabla(\Delta u(t))) \cdot (\nabla(\Delta u(t))) && \text{(by Green's identity (1.6.2))} \\ &= -\|\nabla(\Delta u(t))\|_{L^2}^2 \end{aligned}$$

For $0 < \epsilon < T < \infty$, integrating on $[\epsilon, T]$, we get

$$\varphi(T) - \varphi(\epsilon) = -\int_0^T \|\nabla(\Delta u(t))\|_{L^2}^2 dt. \tag{3.1.25}$$

Note that from (3.1.24), we have $u \in C([0,\infty); H_2)$. That means $\|u(t)-u(t_0)\|_{H_2} \to 0$ whenever $|t-t_0| \to 0$ for any $t_0 \in [0,\infty)$. Now,

$$\|u(t)-u(t_0)\|_{H_2}^2 = \|\Delta(u(t)-u(t_0))\|_{L^2}^2 + \|u(t)-u(t_0)\|_{L^2}^2 \, .$$

Thus,

$$\begin{aligned} &\|u(t)-u(t_0)\|_{H_2} \to 0 \\ \Rightarrow\ &\|\Delta u(t) - \Delta u(t_0)\|_{L^2} \to 0 \text{ for any } t_0 \in [0,\infty). \end{aligned}$$

Therefore, $\Delta u \in C([0,\infty); L^2(\Omega))$ and hence, $\varphi(\epsilon) \to \varphi(0)$ as $\epsilon \to 0$. Taking the limit as $\epsilon \to 0$ in (3.1.25), we get

$$\varphi(T) - \varphi(0) = -\int_0^T \|\nabla(\Delta u(t))\|_{L^2}^2 dt. \tag{3.1.26}$$

Note that

$$\|\Delta u(t)\|_{H^1}^2 = \|\nabla(\Delta u(t))\|_{L^2}^2 + \|\Delta u(t)\|_{L^2}^2.$$

Then from (3.1.26) and (3.1.5), we have for any $T > 0$,

$$\begin{aligned} \int_0^T \|\Delta u(t)\|_{H^1}^2 dt &= \frac{1}{2}\|\nabla u_0\|_{L^2}^2 + \frac{1}{2}\|\Delta u_0\|_{L^2}^2 - \frac{1}{2}\left(\|\nabla u(T)\|_{L^2}^2 + \|\Delta u(T)\|_{L^2}^2\right) \\ &\le \frac{1}{2}\|\nabla u_0\|_{L^2}^2 + \frac{1}{2}\|\Delta u_0\|_{L^2}^2. \end{aligned}$$

Therefore,

$$\frac{\partial u}{\partial t} = \Delta u \in L^2\left(0,\infty; H^1(\Omega)\right).$$

Recall that by Remark 1.7.2, we have

$$\|u(t)\|_{H^3}^2 \le 2C^2(\|u(t)\|_{H^1}^2 + \|\Delta u(t)\|_{H^1}^2).$$

Therefore, for any $T > 0$,

$$\begin{aligned} &\int_0^T \|u(t)\|_{H^3}^2 dt \le 2C^2\left(\int_0^T \|u(t)\|_{H^1}^2 dt + \int_0^T \|\Delta u(t)\|_{H^1}^2 dt\right) \\ &\le 2C^2\left(\int_0^T \|u(t)\|_{H^2}^2 dt + \int_0^T \|\Delta u(t)\|_{H^1}^2 dt\right) \\ &< \infty. \end{aligned}$$

The last inequality holds because $u \in L^2(0,\infty; H^2(\Omega))$ as shown in the proof of Theorem 3.1.2(i) and $\Delta u \in L^2(0,\infty; H^1(\Omega))$ as shown above. Thus, $u \in L^2(0,\infty; H^3(\Omega))$. This concludes the proof of (ii).

■

Proof of (iii). Consider the maximal monotone operator $A : D(A) \subset H \to H$ given by

$$\begin{cases} H = L^2(\Omega), \\ D(A) = H^2(\Omega) \cap H_0^1(\Omega), \\ Au = -\Delta u. \end{cases}$$

Then by Theorem 2.2.1, given $u_0 \in D(A^k)$, there exists a unique solution u of

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + Au = 0 \text{ on } [0,\infty), \\ u(0) = u_0 \end{cases}$$

that satisfies

$$u \in C^{k-j}([0,\infty); D(A^j)) \ \ \forall\, j \in \{0, 1, \cdots, k\}.$$

Recall from the proof of Theorem 3.1.1, that

$$D(A^k) = \{u \in H^{2k}(\Omega) | u = \Delta u = \cdots = \Delta^{k-1} u = 0 \text{ on } \Gamma\}.$$

Then from the assumptions for (iii), it follows that $u_0 \in D(A^k)$ for all $k \in \mathbb{N}$ and hence,

$$u \in C^{k-j}([0,\infty); D(A^j)) \ \ \forall\, j \in \{0, 1, \cdots, k\}, \ \forall k \in \mathbb{N}.$$

Following a similar argument as in the proof of Theorem 3.1.1, we conclude that $u \in C^\infty(\overline{\Omega} \times [0,\infty))$. ■

<u>*Remark 3.1.1:*</u> The compatibility conditions $u_0 \in H^k(\Omega)$ for all $k \in \mathbb{N}$; $\Delta^j u_0 = 0$ for all $j \in \mathbb{N}$ are, in fact, necessary conditions for (3.1.1a-c) to have a solution that is smooth upto $t = 0$, that is, $u \in C^\infty(\overline{\Omega} \times [0,\infty))$.

To see this suppose the solution u of (3.1.1a-c) satisfies $u \in C^\infty(\overline{\Omega} \times [0,\infty))$. That is, u has continuous partial derivatives of any order. So $\frac{\partial^j u}{\partial t^j}$ is continuous on $\overline{\Omega} \times [0,\infty)$ for all $j \in \mathbb{N}$. Note that $u = 0$ on $\Gamma \times (0,\infty)$. Therefore, $\frac{\partial^j u}{\partial t^j} = 0$ on $\Gamma \times [0,\infty)$. By continuity of $\frac{\partial^j u}{\partial t^j}$, we have

$$\frac{\partial^j u}{\partial t^j} = 0 \text{ on } \Gamma \times [0,\infty) \ \ \forall j \in \mathbb{N}. \tag{3.1.27}$$

Again,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}(\Delta u) = \Delta \left(\frac{\partial u}{\partial t} \right) = \Delta(\Delta u) = \Delta^2 u$$

in $\Omega \times (0,\infty)$. By induction, we conclude that $\frac{\partial^j u}{\partial t^j} = \Delta^j u$ in $\Omega \times (0,\infty)$ for all $j \in \mathbb{N}$. Again, by continuity, $\frac{\partial^j u}{\partial t^j} = \Delta^j u$ in $\overline{\Omega} \times [0,\infty)$. That also means $\frac{\partial^j u}{\partial t^j} = \Delta^j u$ in $\Gamma \times [0,\infty)$. And from (3.1.27), it follows, for all $j \in \mathbb{N}$,

$$\Delta^j u = 0 \text{ in } \Gamma \times \{0\}.$$

That is,

$$\Delta^j u_0 = 0 \text{ on } \Gamma \quad \forall j \in \mathbb{N}.$$

3.2 Wave Equation

In this section we will apply the Hille-Yosida Theorem to the wave equation. The wave equation is a second-order partial differential equation used for the description of waves.

Let $\Omega \subset \mathbb{R}^N$ be an open set with boundary $\Gamma = \partial\Omega$. We are concerned with finding a function $u(x,t) : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ that satisfies:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0 & \text{in } \Omega \times (0,\infty), & (3.2.1a) \\ u = 0 & \text{on } \Gamma \times (0,\infty), & (3.2.1b) \\ u(x,0) = u_0(x) & \text{on } \Omega, & (3.2.1c) \\ \dfrac{\partial u}{\partial t}(x,0) = v_0(x) & \text{on } \Omega, & (3.2.1d) \end{cases}$$

where Δu denotes the Laplacian of u in the spatial variables $x_1, \cdots, x_N$, that is, $\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$, u_0 and v_0 are given functions. $\left(\frac{\partial^2}{\partial t^2} - \Delta\right)$ is called the d'Alembertian and often denoted by $\Box$.

For $N = 1$, and $\Omega = (0,1)$, $\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$ models small vibrations of a string, for $N = 2$, it models small vibrations of an elastic membranes. In general, it models the propagation of a wave (acoustic, electromagnetic etc) in a homogeneous elastic medium. At any instant of time t, the graph of the function $t \mapsto u(x,t)$ represents the configuration of the wave at that instant.

(3.2.1b) is the Dirichlet boundary condition that signifies that the string or memberane is fixed on Γ. (3.2.1c) and (3.2.1d) provide the initial conditions and are referred to as Cauchy data. u_0, v_0 represent the initial configuration and initial velocity respectively.

For simplicity, we will assume throughout that Ω is of class C^∞ and that the boundary $\Gamma = \partial\Omega$ is bounded. In order to apply the Hille-Yosida theorem, we will consider the function $t \mapsto u(x,t)$ represented by $u(t)$ in an appropriate function space.

Now, we will state and prove the following results regarding the existence, uniqueness and regularity of solutions of the wave equation.

Theorem 3.2.1 (existence and uniqueness)**.** *Suppose* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ *and* $v_0 \in H_0^1(\Omega)$*, Then there exists a unique solution* u *of (3.2.1a-d) satisfying:*

$$u \in C([0,\infty); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,\infty); H_0^1(\Omega)) \cap C^2([0,\infty); L^2(\Omega)). \quad (3.2.2)$$

Furthermore,

$$\left\| \frac{\partial u}{\partial t}(t) \right\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 = \|v_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2, \ \forall t \geq 0. \quad (3.2.3)$$

(Here, we use the notation $\left\|\frac{\partial u}{\partial t}(t)\right\|_{L^2}^2 = \int_\Omega \left|\frac{\partial u}{\partial t}(t)\right|^2 dx$; $\|\nabla u(t)\|_{L^2}^2 = \int_\Omega \nabla u(t) \cdot \nabla u(t) dx$*)*

Theorem 3.2.2 (regularity)**.** *If we further assume that the initial data* u_0, v_0 *satisfies*

$$u_0, v_0 \in H^k(\Omega) \ \forall k \in \mathbb{N}$$

and the compatibility conditions

$$\Delta^j u_0 = \Delta^j v_0 = 0 \quad on \ \Gamma, \ \forall j \in \mathbb{N}$$

then the solution u *of (3.2.1a-d) satisfies*

$$u \in C^\infty(\overline{\Omega} \times [0,\infty)).$$

Proof. We can write (3.2.1) as a system of first order differential equations as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = v & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = \Delta u & \text{in } \Omega \times (0,\infty). \end{cases} \quad (3.2.4)$$

Write $U = \begin{pmatrix} u \\ v \end{pmatrix}$, then (3.2.4) becomes

$$\begin{aligned} \frac{\mathrm{d}U}{\mathrm{d}t} = \begin{pmatrix} v \\ \Delta u \end{pmatrix} &= -\begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ &= -AU \end{aligned}$$

where $A = \begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix}$. Now, consider the Hilbert space $H = H_0^1(\Omega) \times L^2(\Omega)$ equipped with the scalar product

$$(U_1, U_2)_H = \int_\Omega \nabla u_1 \cdot \nabla u_2 dx + \int_\Omega u_1 u_2 dx + \int_\Omega v_1 v_2 dx$$

where $U_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}$. Note that $\|U\|_H^2 = \|u\|_{H^1}^2 + \|v\|_{L^2}^2$, where $U = \begin{pmatrix} u \\ v \end{pmatrix}$.

Take the unbounded linear operator A on H,

$$A : D(A) \subset H \to H$$

defined by

$$\begin{cases} D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \\ AU = \begin{pmatrix} -v \\ -\Delta u \end{pmatrix}, \text{ whenever } U = \begin{pmatrix} u \\ v \end{pmatrix}. \end{cases}$$

Now, we will show that $(A+I)$ is maximal montone. For $U = \begin{pmatrix} u \\ v \end{pmatrix}$, we have

$$\begin{aligned} & ((A+I)U, U)_H \\ &= (AU, U)_H + (U, U)_H \\ &= \int_\Omega \nabla(-v) \cdot \nabla u + \int_\Omega -vu + \int_\Omega (-\Delta u)v + \|U\|_H^2 \\ &= -\int_\Omega vu + \int_\Omega u^2 + \int_\Omega |\nabla u|^2 + \int_\Omega v^2 \qquad \text{(using Green's identity (1.6.2))} \\ &\geq \int_\Omega |\nabla u|^2 + \int_\Omega (u^2+v^2) - \int_\Omega \frac{u^2+v^2}{2} \qquad \text{(using } -vu \geq -(u^2+v^2)/2) \\ &= \int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega (u^2+v^2) \\ &\geq 0. \end{aligned}$$

Hence, $A+I$ is monotone. Now, in order to show that $A+I$ is maximal monotone, we will show that $R(A+2I) = H$. That is, given $F = \begin{pmatrix} f \\ g \end{pmatrix}$, we want to find $U = \begin{pmatrix} u \\ v \end{pmatrix} \in D(A)$ such that $(A+2I)U = F$. Then we have the system, of equations:

$$\begin{cases} -v + 2u = f, & \text{on } \Omega \\ -\Delta u + 2v = g, & \text{on } \Omega \end{cases} \tag{3.2.5}$$

with $u \in H^2(\Omega) \cap H_0^1(\Omega), v \in H_0^1(\Omega)$. Putting $v = 2u - f$, (3.2.5) becomes

$$-\Delta u + 4u = 2f + g. \tag{3.2.6}$$

By Remark 1.7.1, there exists a unique solution u of (3.2.6). Taking $v = 2u - f$, we have a solution $U = \begin{pmatrix} u \\ v \end{pmatrix} \in D(A)$ such that $(A+2I)U = F$. That is, $R(A+2I) = H$, and hence, $A+I$ is maximal monotone.

Now, denote $A_1 = A + I$. Note that $D(A_1) = D(A)$. Then by Theorem 2.1.5, given $U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in D(A_1) = D(A)$, there exists a unique solution, V of

$$\begin{cases} \dfrac{\mathrm{d}V}{\mathrm{d}t} + A_1 V = 0 \quad \text{on } [0, \infty), \\ V(0) = U_0, \end{cases}$$

satisfying

$$V \in C^1([0, \infty); H) \cap C([0, \infty); D(A)).$$

Now, let $U(t) = e^t V(t)$, Then for $t \in [0, \infty)$,

$$\begin{aligned} \frac{\mathrm{d}U}{\mathrm{d}t}(t) &= e^t \frac{\mathrm{d}V}{\mathrm{d}t}(t) + e^t V(t) \\ &= e^t \left[\frac{\mathrm{d}V}{\mathrm{d}t}(t) + V(t) \right] \\ &= e^t [-AV(t)] \\ &= -A(e^t V(t)) \\ &= -AU(t). \end{aligned}$$

That is, $\frac{\mathrm{d}U}{\mathrm{d}t} + AU = 0$ on $[0, \infty)$. Also, $U(0) = V(0) = U_0$. Thus $U(t) = e^t V(t)$ is the unique solution to

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} + AU = 0 \;\; \text{on } [0, \infty), \\ U(0) = U_0, \end{cases}$$

with

$$U \in C^1([0, \infty); H) \cap C([0, \infty); D(A))$$

and (3.2.2) follows.

Now, note that

$$\|\nabla u\|_{L^2}^2 = \int_\Omega \nabla u \cdot \nabla u \; dx = \int_\Omega \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^2 dx = \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2}^2 .$$

Also, note that

$$\begin{aligned} \frac{\partial}{\partial t} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2}^2 &= 2 \left(\frac{\partial u}{\partial x_i}, \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x_i} \right) \right)_{L^2} = 2 \left(\frac{\partial u}{\partial x_i}, \frac{\partial}{\partial x_i} \left(\frac{\partial u}{\partial t} \right) \right)_{L^2} \\ &= 2 \int_\Omega \left(\frac{\partial u}{\partial x_i} \right) \left(\frac{\partial}{\partial x_i} \left(\frac{\partial u}{\partial t} \right) \right) \; dx. \end{aligned}$$

Therefore,

$$\frac{\partial}{\partial t}\|\nabla u(t)\|_{L^2}^2 = 2\int_\Omega \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}\right)\left(\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial t}\right)\right) dx = 2\int_\Omega \nabla u \cdot \nabla\left(\frac{\partial u}{\partial t}\right) dx.$$

Note that $u(t) \in D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and

$$\begin{aligned} \frac{\partial u}{\partial t}(t) &= v(t) \in H_0^1(\Omega) \\ \Rightarrow \frac{\partial u}{\partial t}(t) &= 0 \text{ on } \Gamma. \end{aligned}$$

By Green's identity (1.6.2), we have

$$\frac{\partial}{\partial t}\|\nabla u\|_{L^2}^2 = -\int_\Omega (\Delta u)\left(\frac{\partial u}{\partial t}\right) dx. \tag{3.2.7}$$

Again,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\|v\|_{L^2}^2\right) = \left(v, \frac{\partial v}{\partial t}\right)_{L^2} = \int_\Omega \left(\frac{\partial u}{\partial t}\right)\left(\frac{\partial^2 u}{\partial t^2}\right) dx. \tag{3.2.8}$$

Adding (3.2.7) and (3.2.8),

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{1}{2}\|v\|_{L^2}^2 + \frac{1}{2}\|\nabla u\|_{L^2}^2\right) &= \int_\Omega \left(\frac{\partial u}{\partial t}\right)\left(\frac{\partial^2 u}{\partial t^2}\right) dx - \int_\Omega (\Delta u)\left(\frac{\partial u}{\partial t}\right) \\ &= \int_\Omega \frac{\partial u}{\partial t}\left(\frac{\partial^2 u}{\partial t^2} - \Delta u\right) dx \\ &= 0. \qquad \left(\text{since } \tfrac{\partial^2 u}{\partial t^2} = \Delta u\right) \end{aligned}$$

Integrating wrt t,

$$\begin{aligned} &\int_0^T \frac{\partial}{\partial t}\left(\frac{1}{2}\left\|\frac{\partial u}{\partial t}(t)\right\|_{L^2}^2 + \frac{1}{2}\|\nabla u(t)\|_{L^2}^2\right) dt = 0 \\ &\Rightarrow \frac{1}{2}\left\|\frac{\partial u}{\partial t}(T)\right\|_{L^2}^2 + \frac{1}{2}\|\nabla u(T)\|_{L^2}^2 = \frac{1}{2}\|v_0\|_{L^2}^2 + \frac{1}{2}\|\nabla u_0\|_{L^2}^2 \\ &\Rightarrow \left\|\frac{\partial u}{\partial t}(T)\right\|_{L^2}^2 + \|\nabla u(T)\|_{L^2}^2 = \|v_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2. \end{aligned}$$

That is, (3.2.3) holds. This concludes the proof of Theorem 3.2.1.

■

Proof of Theorem 3.2.2. Denote

$$D_k = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \middle| \begin{matrix} u \in H^{k+1}(\Omega),\ \Delta^j u = 0 \ \text{ on } \ \Gamma,\ \text{for } 0 \le j \le \left[\frac{k}{2}\right] \\ v \in H^k(\Omega),\ \Delta^j v = 0 \ \text{ on } \ \Gamma,\ \text{for } , 0 \le j \le \left[\frac{k-1}{2}\right] \end{matrix} \right\}.$$

We claim that $D(A^k) = D_k$ for all integers $k \geq 1$. Indeed for $k = 1$, we have

$$\begin{aligned} D(A) &= (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \\ &= \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \middle| \begin{matrix} u \in H^2(\Omega), \ u = 0 \ \text{ on } \ \Gamma \\ v \in H^1(\Omega), \ v = 0 \ \text{ on } \ \Gamma \end{matrix} \right\} \\ &= D_1. \end{aligned}$$

Suppose $D(A^{k-1}) = D_k$ (our induction hypothesis). Now, recall that

$$\begin{aligned} D(A^k) &= \{U \in D(A^{k-1}) | AU \in D(A^{k-1})\} \\ &= \{U \in D_{k-1} | AU \in D_{k-1}\}. \qquad \text{(by induction hypothesis)} \end{aligned}$$

From $U \in D_{k-1}$, we have

$$u \in H^k(\Omega), \Delta^j u = 0 \text{ on } \Gamma \text{ for } 0 \leq j \leq \left[\frac{k-1}{2}\right], \tag{3.2.10}$$

$$v \in H^{k-1}(\Omega), \Delta^j v = 0 \text{ on } \Gamma \text{ for } 0 \leq j \leq \left[\frac{k-2}{2}\right]. \tag{3.2.11}$$

And from $AU \in D_{k-1}$, we have

$$v \in H^k(\Omega), \Delta^j v = 0 \text{ on } \Gamma \text{ for } 0 \leq j \leq \left[\frac{k-1}{2}\right], \tag{3.2.12}$$

$$\Delta u \in H^{k-1}(\Omega), \Delta^{j+1} u = 0 \text{ on } \Gamma \text{ for } 0 \leq j \leq \left[\frac{k-2}{2}\right]. \tag{3.2.13}$$

From (3.2.13) and (3.2.10), we have

$$-\Delta u + u \in H^{k-1}(\Omega), \tag{3.2.14}$$

$$\Delta^j u = 0 \text{ on } \Gamma \text{ for } 0 \leq j \leq \left[\frac{k-1}{2}\right]. \tag{3.2.15}$$

Using Theorem 1.7.1 and Theorem 1.7.2, from (3.2.14), we have $u \in H^{k+1}(\Omega)$. This together with (3.2.15) and (3.2.12) implies that $U \in D_k$ whenever $U \in D(A^k)$. That is, $D(A^k) \subset D_k$. It can be easily checked that $U = \begin{pmatrix} u \\ v \end{pmatrix} \in D_k \Rightarrow \ U \in D_{k-1}, AU \in D_{k-1}$. That is,

$$\begin{aligned} D_k &\subseteq \{U \in D_{k-1} | AU \in D_{k-1}\} = \{U \in D(A^{k-1}) | AU \in D(A^{k-1})\} \\ &= D(A^k). \qquad \text{(by induction hypothesis)} \end{aligned}$$

Hence, we can conclude $D_k = D(A^k)$. Thus, by induction, it holds that $D(A^k) = D_k$ for all integers $k \geq 1$. In particular, we have $D(A^k) \subset H^{k+1}(\Omega) \times H^k(\Omega)$. We claim that the above injection is continuous.

For $k=1$, $D(A) = (H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$. Recall that

$$\|U\|_{D(A)}^2 = \|U\|_H^2 + \|AU\|_H^2.$$

Now,

$$\|U\|_H^2 = \|u\|_{H^1}^2 + \|v\|_{L^2}^2, \tag{3.2.16}$$
$$\|AU\|_H^2 = \|v\|_{H^1}^2 + \|\Delta u\|_{L^2}^2. \tag{3.2.17}$$

Recall that by Remark 1.7.2,

$$\begin{aligned}\|u\|_{H^2} &\le 2C^2(\|\Delta u\|_{L^2}^2 + \|u\|_{L^2}^2)\\ &\le 2C^2(\|\Delta u\|_{L^2}^2 + \|u\|_{H^1}^2).\end{aligned} \tag{3.2.18}$$

Therefore,

$$\begin{aligned}&\|U\|_{D(A)} \to 0\\ \Rightarrow\ &\|u\|_{H^1}, \|\Delta u\|_{L^2}, \|v\|_{H^1} \to 0 && \text{(from (3.2.16) and (3.2.17))}\\ \Rightarrow\ &\|u\|_{H^2}^2 + \|v\|_{H^1}^2 \to 0 && \text{(from (3.2.18))}\\ \Rightarrow\ &\|U\|_{H^2\times H^1} \to 0.\end{aligned}$$

Thus, $D(A)\subset H^2(\Omega)\times H^1(\Omega)$ is a continuous injection.
Suppose now $D(A^{k-1})\subset H^k(\Omega)\times H^{k-1}(\Omega)$ (our induction hypothesis). Note that

$$\begin{aligned}\|U\|_{D(A^k)}^2 &= \sum_{j=0}^{k}\|A^jU\|_H^2\\ &= \|U\|_{D(A^{k-1})}^2 + \|A^kU\|_H^2 &&(3.2.19)\\ &= \|U\|_H^2 + \|AU\|_{D(A^{k-1})}^2. &&(3.2.20)\end{aligned}$$

Therefore,

$$\begin{aligned}&\|U\|_{D(A^k)} \to 0\\ \Rightarrow\ &\|U\|_{D(A^{k-1})} \to 0 && \text{(from (3.2.19))}\\ \Rightarrow\ &\|U\|_{H^k\times H^{k-1}} \to 0 && \text{(from induction hypothesis)}\\ \Rightarrow\ &\|u\|_{H^k}, \|v\|_{H^{k-1}} \to 0. && (3.2.21)\end{aligned}$$

Again,

$$\begin{aligned}&\|U\|_{D(A^k)} \to 0\\ \Rightarrow\ &\|AU\|_{D(A^{k-1})} \to 0 && \text{(from (3.2.20))}\\ \Rightarrow\ &\|AU\|_{H^k\times H^{k-1}} \to 0 && \text{(by induction hypothesis)}\\ \Rightarrow\ &\|v\|_{H^k}, \|\Delta u\|_{H^{k-1}} \to 0. && (3.2.22)\end{aligned}$$

Recall that by Remark 1.7.2

$$\begin{aligned}\|u\|^2_{H^{k+1}} &\leq 2C^2(\|\Delta u\|^2_{H^{k-1}} + \|u\|^2_{H^{k-1}}) \\ &\leq 2C^2(\|\Delta u\|^2_{H^{k-1}} + \|u\|^2_{H^k}). \end{aligned} \tag{3.2.23}$$

Therefore,

$$\begin{aligned} &\|U\|_{D(A^k)} \to 0 \\ \Rightarrow\ &\|u\|_{H^{k+1}}, \|v\|_{H^k} \to 0 && \text{(from 3.2.21,3.2.22,3.2.23)} \\ \Rightarrow\ &\|U\|_{H^{k+1}\times H^k} \to 0. && (\text{since } \|U\|^2_{H^{k+1}\times H^k} = \|u\|^2_{H^{k+1}} + \|v\|^2_{H^k}) \end{aligned}$$

Thus, by induction $D(A^k) \subset H^{k+1}(\Omega) \times H^k(\Omega)$ is a continuous injection for all integers $k \geq 1$.

Now, note that $D(A^k) = D(A_1^k)$, where $A_1 = A + I$. It can be easily checked that $\|\cdot\|_{D(A^k)}$ and $\|\cdot\|_{D(A_1^k)}$ are equivalent norms in $D(A^k) = D(A_1^k)$. Now given $U_0 \in D(A^k)$, by Theorem 2.2.1, we have that the unique solution, V of the problem

$$\begin{cases} \dfrac{\mathrm{d}V}{\mathrm{d}t} + A_1 V = 0 & \text{on } [0, \infty), \\ V(0) = U_0 \end{cases}$$

satisfies $V \in C^{k-j}([0,\infty); D(A_1^j)) \quad \forall\, j \in \{0, 1, \cdots, k\}$. That is,

$$V \in C^{k-j}([0,\infty); D(A^j)) \quad \forall\, j \in \{0, 1, \cdots, k\}.$$

Then the unique solution $U(t) = e^t V(t)$ of the problem

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} + AU = 0 & \text{on } [0, \infty), \\ U(0) = U_0 \end{cases}$$

satisfies

$$U \in C^{k-j}([0,\infty); D(A^j)) \quad \forall j \in \{0, 1, \cdots, k\}. \tag{3.2.24}$$

Now, since by the assumptions on the initial data it follows that $U_0 \in D(A^k) \quad \forall k \in \mathbb{N}$, (3.2.24) holds for all $k \in \mathbb{N}$.

Note that from Corollary 1.5.3, given any $l \in \mathbb{N}$, $\exists$ an integer $m > l$ such that $H^m(\Omega) \subset C^l(\overline{\Omega})$ is a continuous injection. Then clearly, so is $H^{m+1}(\Omega) \subset H^m(\Omega) \subset C^l(\overline{\Omega})$. Choose $k = l + m, j = m$. Then (3.2.24) reads

$$\begin{aligned} &U \in C^l\left([0,\infty); H^{m+1}(\Omega) \times H^m(\Omega)\right) \\ \Rightarrow\ &U \in C^l([0,\infty); C^l(\overline{\Omega}) \times C^l(\overline{\Omega})). \end{aligned} \tag{3.2.25}$$

Since (3.2.25) holds for any $l \in \mathbb{N}$, it follows

$$U \in C^{\infty}([0,\infty); C^{\infty}(\overline{\Omega}) \times C^{\infty}(\overline{\Omega}))$$

That is,

$$u \in C^{\infty}(\overline{\Omega} \times [0,\infty)).$$

This concludes the proof of Theorem 3.2.2. ■

3.3 Linearized equations of coupled sound and heat flow

In this section we apply the Hille-Yosida Theorem to show the existence, uniqueness and regularity of solutions to the initial value problem for the linearized equations of coupled sound and heat flow. We follow the same argument presented by Matsubara and Yokota in [12] but with some modified computations. We assume that Ω is a fixed domain in $\mathbb{R}^N$ and that $\Gamma := \partial\Omega$ is bounded and smooth.

Formulation of the problem

The problem arises from the infinitesimal motions of a compressible fluid. In such a fluid there can be a significant difference in temperature from one point to another and hence, the effect of transfer of energy by thermal conduction cannot be ignored. As formulated by Carasso in [3], the linearized equations for the conservation of mass, momentum and energy are (c.f. [7], [15]) as follows:

$$\begin{cases} \dfrac{\partial w}{\partial t} = c\nabla \cdot \vec{u} \\ \dfrac{\partial \vec{u}}{\partial t} = c\nabla w - c\nabla e \\ \dfrac{\partial e}{\partial t} = \sigma\Delta e - (\gamma - 1)c\nabla \cdot \vec{u} \end{cases}$$

where c is the isothermal sound speed, $\gamma > 1$ ia the ratio of specific heats and $\sigma > 0$ is the thermal conductivity. For simplicity it is further assumed that the disturbance is confined to the fixed domain Ω and that the ambient conditions prevail on the boundary Γ. This ensures $e = w = 0$ on Γ, $\forall\, t \geq 0$.

Taking the divergence of the second equation and eliminating $\nabla \cdot \vec{u}$ from the system, we are left with the following equations for the unknown scalar fields, $w(x,t)$

and $e(x,t)$.

$$\begin{cases} w_{tt} = c^2\Delta w - c^2\Delta e \quad \text{in} \;\; \Omega \times (0,\infty), \\ e_t = \sigma\Delta e - (\gamma-1)w_t \quad \text{in} \;\; \Omega \times (0,\infty). \end{cases} \tag{3.3.1}$$

Consider the following system of equations:

$$\begin{cases} w_{tt} = c^2\Delta w - c^2\Delta e + m^2 w \quad \text{in} \;\; \Omega \times (0,\infty), \\ e_t = \sigma\Delta e - (\gamma-1)w_t \quad \text{in} \;\; \Omega \times (0,\infty), \\ e = w = 0 \quad \text{on} \;\; \Gamma \times [0,\infty), \\ w(x,0) = w_0(x), w_t(x,0) = v_0(x), e(x,0) = e_0(x) \quad \text{on} \;\; \Omega, \end{cases} \tag{3.3.2}$$

where $\sigma > 0$, $\gamma > 1$, $c > 0$ and $m \in \mathbb{R}$. We see that (3.3.2) is reduced to (3.3.1) for $m = 0$. Carasso ([3]) analyzed the problem for the case with $m = 0$ by a least squares procedure. Here, as in [12], we apply Hille-Yosida Theorem to the problem (3.3.2) with $m \in \mathbb{R}$. We will state and prove the following results about the solution to the system (3.3.2).

Theorem 3.3.1 (existence and uniqueness)**.** *Suppose $w_0, e_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $v_0 \in H_0^1(\Omega)$. Then there exists a unique solution (w,e) of (3.3.2) satisfying:*

$$w \in C([0,\infty); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,\infty); H_0^1(\Omega)) \cap C^2([0,\infty); L^2(\Omega)), \tag{3.3.3}$$
$$e \in C([0,\infty); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,\infty); H_0^1(\Omega)) \cap C^2([0,\infty); L^2(\Omega)). \tag{3.3.4}$$

Furthermore, for some $\alpha > 0$, the following estimates hold:

$$\begin{aligned} &\|w(t)\|_{H^1}^2 + \frac{1}{c^2}\|v(t)\|_{L^2}^2 + \frac{1}{\gamma-1}\|e(t)\|_{H^1}^2 \\ &\leq e^{2\alpha t}\left(\|w_0\|_{H^1}^2 + \frac{1}{c^2}\|v_0\|_{L^2}^2 + \frac{1}{\gamma-1}\|e_0\|_{H^1}^2\right) \quad \forall\, t \geq 0, \end{aligned} \tag{3.3.5}$$

$$\begin{aligned} &\|v(t)\|_{H^1}^2 + \frac{1}{c^2}\left\|c^2\Delta w(t) + m^2 w(t) - c^2\Delta e(t)\right\|_{L^2}^2 + \frac{1}{\gamma-1}\|(\gamma-1)v(t) - \sigma\Delta e(t)\|_{H^1}^2 \\ &\leq e^{2\alpha t}\Bigg(\|v_0\|_{H^1}^2 + \frac{1}{c^2}\left\|c^2\Delta w_0 + m^2 w_0 - c^2\Delta e_0\right\|_{L^2}^2 \\ &\qquad + \frac{1}{\gamma-1}\|(\gamma-1)v_0 - \sigma\Delta e_0\|_{H^1}^2\Bigg) \quad \forall\, t \geq 0. \end{aligned} \tag{3.3.6}$$

Theorem 3.3.2 (regularity)**.** *Suppose the initial data satisfy* $w_0, v_0, e_0 \in H^k(\Omega)\ \forall\ k \in \mathbb{N}$ *and the following compatibilty conditions hold:*

$$\begin{aligned} \Delta^j w_0 = 0 \quad & on \quad \Gamma,\ \forall\ j \in \mathbb{N}, \\ \Delta^j v_0 = 0 \quad & on \quad \Gamma,\ \forall\ j \in \mathbb{N}, \\ \Delta^j e_0 = 0 \quad & on \quad \Gamma,\ \forall\ j \in \mathbb{N}. \end{aligned}$$

Then the solution (w, e) *of (3.3.2) obtained in Theorem 3.3.1 satisfies*

$$(w, e) \in C^\infty(\overline{\Omega} \times [0, \infty)) \times C^\infty(\overline{\Omega} \times [0, \infty)).$$

Existence and uniqueness of solution

Proof of Theorem 3.3.1. We can write (3.3.2) as a system of first order equations as follows:

$$\begin{cases} w_t = v, \\ v_t = c^2\Delta w + m^2 w - c^2\Delta e, \\ e_t = \sigma\Delta e - (\gamma - 1)v. \end{cases} \tag{3.3.7}$$

If we denote $U = \begin{pmatrix} w \\ v \\ e \end{pmatrix}$, then (3.3.7) becomes

$$\begin{aligned} \frac{\mathrm{d}U}{\mathrm{d}t} &= \begin{pmatrix} v \\ c^2\Delta w + m^2 w - c^2\Delta e \\ \sigma\Delta e - (\gamma - 1)v \end{pmatrix} \\ &= -\begin{pmatrix} -v \\ -c^2\Delta w - m^2 w + c^2\Delta e \\ (\gamma - 1)v - \sigma\Delta e \end{pmatrix} \\ &= -\begin{pmatrix} 0 & -I & 0 \\ -(c^2\Delta + m^2 I) & 0 & c^2\Delta \\ 0 & (\gamma - 1)I & -\sigma\Delta \end{pmatrix} \begin{pmatrix} w \\ v \\ e \end{pmatrix}. \end{aligned} \tag{3.3.8}$$

If we denote the operator $A = \begin{pmatrix} 0 & -I & 0 \\ -(c^2\Delta + m^2 I) & 0 & c^2\Delta \\ 0 & (\gamma - 1)I & -\sigma\Delta \end{pmatrix}$, then (3.3.8) becomes

$$\frac{\mathrm{d}U}{\mathrm{d}t} = -AU$$

Then we can rewrite (3.3.2) as the problem

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} + AU = 0 \quad \text{on} \quad (0,\infty), \\ U(0) = U_0. \end{cases}$$

Now, let us consider the operator $A : D(A) \subset H \to H$, where $D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))$; $H = H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega)$ is a Hilbert space equipped with the scalar product

$$(U_1, U_2) = \int_\Omega \nabla w_1 \cdot \nabla w_2 + \int_\Omega w_1 w_2 + \frac{1}{c^2}\int_\Omega v_1 v_2 + \frac{1}{\gamma-1}\int_\Omega \nabla e_1 \cdot \nabla e_2 + \frac{1}{\gamma-1}\int_\Omega e_1 e_2$$

where $U_i = \begin{pmatrix} w_i \\ v_i \\ e_i \end{pmatrix}$. That is,

$$(U_1, U_2) = (w_1, w_2)_{H^1} + \frac{1}{c^2}(v_1, v_2)_{L^2} + \frac{1}{\gamma-1}(e_1, e_2)_{H^1}.$$

So, the corresponding norm in H is given by

$$\|U\|^2 = \|w\|_{H^1}^2 + \frac{1}{c^2}\|v\|_{L^2}^2 + \frac{1}{\gamma-1}\|e\|_{H^1}^2.$$

Before we proceed further, we first state and prove the following two claims.

Claim 1. $A + \alpha I$ *is a monotone operator for any positive constant,*
$\alpha > \max\left\{|m|, \left(\frac{1}{2} + \frac{m^2}{2c^2}\right), c^2\left(\frac{1}{2} + \frac{m^2}{2c^2}\right), \frac{\gamma-1}{2}\right\}$.

Proof of Claim 1. Note that

$$\begin{aligned}(AU, U) = {} & \int_\Omega \nabla(-v)\cdot\nabla w + \int_\Omega -vw + \frac{1}{c^2}\int_\Omega(-c^2\Delta w - m^2 w + c^2\Delta e)v \\ & + \frac{1}{\gamma-1}\int_\Omega \nabla((\gamma-1)v - \sigma\Delta e)\cdot\nabla e + \frac{1}{\gamma-1}\int_\Omega((\gamma-1)v - \sigma\Delta e)e. \quad (3.3.9)\end{aligned}$$

By Green's identity (1.6.2), we have

$$\int_\Omega \nabla(-v)\cdot\nabla w = \int_\Omega v\Delta w, \quad (3.3.10)$$

$$\frac{1}{\gamma-1}\int_\Omega \nabla((\gamma-1)v - \sigma\Delta e) = -\frac{1}{\gamma-1}\int_\Omega((\gamma-1)v - \sigma\Delta e)\Delta e. \quad (3.3.11)$$

From (3.3.9), (3.3.10) and (3.3.11), we have

$$\begin{aligned}
&(AU, U)\\
&= \int_\Omega v\Delta w - \int_\Omega vw + \frac{1}{c^2}\int_\Omega (-c^2\Delta w - m^2 w + c^2\Delta e)v\\
&\quad - \frac{1}{\gamma-1}\int_\Omega ((\gamma-1)v - \sigma\Delta e)\Delta e + \frac{1}{\gamma-1}\int_\Omega ((\gamma-1)v - \sigma\Delta e)e\\
&= \int_\Omega v\Delta w - \int_\Omega vw - \int_\Omega v\Delta w - \frac{m^2}{c^2}\int_\Omega vw + \int_\Omega v\Delta e\\
&\quad + \frac{\sigma}{\gamma-1}\int_\Omega |\Delta e|^2 - \int_\Omega v\Delta e - \frac{\sigma}{\gamma-1}\int_\Omega e\Delta e + \int_\Omega ve\\
&= -\left(1+\frac{m^2}{c^2}\right)\int_\Omega vw + \frac{\sigma}{\gamma-1}\int_\Omega |\Delta e|^2 + \int_\Omega ve - \frac{\sigma}{\gamma-1}\int_\Omega e\Delta e\\
&= -\left(1+\frac{m^2}{c^2}\right)\int_\Omega vw + \frac{\sigma}{\gamma-1}\int_\Omega |\Delta e|^2 + \int_\Omega ve + \frac{\sigma}{\gamma-1}\int_\Omega |\nabla e|^2\\
&\hfill \text{(by Green's identity)}\\
&\geq -\left(1+\frac{m^2}{c^2}\right)\int_\Omega vw + \int_\Omega ve. \qquad (3.3.12)
\end{aligned}$$

Note that $vw \leq |v||w| \leq \frac{v^2+w^2}{2}; ve \geq -|v||e| \geq -\frac{v^2+e^2}{2}$, by AM-GM inequality. Therefore, by Theorem (3.3.12),

$$\begin{aligned}
(AU, U) &\geq -\left(\frac{1}{2}+\frac{m^2}{2c^2}\right)\int_\Omega (v^2+w^2) - \frac{1}{2}\int_\Omega (v^2+e^2)\\
&= -\left(1+\frac{m^2}{2c^2}\right)\int_\Omega v^2 - \left(\frac{1}{2}+\frac{m^2}{2c^2}\right)\int_\Omega w^2 - \frac{1}{2}\int_\Omega e^2. \qquad (3.3.13)
\end{aligned}$$

Now take a positive $\alpha > \max\{|m|, (\frac{1}{2}+\frac{m^2}{2c^2}), c^2\ (1+\frac{m^2}{2c^2}\ ), (\gamma-1)/2\}$. Then from (3.3.13), we have

$$\begin{aligned}
&((A+\alpha I)U, U)\\
&\geq -\left(1+\frac{m^2}{2c^2}\right)\int_\Omega v^2 - \left(\frac{1}{2}+\frac{m^2}{2c^2}\right)\int_\Omega w^2 - \frac{1}{2}\int_\Omega e^2 + \alpha\int_\Omega |\nabla w|^2\\
&\quad + \alpha\int_\Omega w^2 + \frac{\alpha}{c^2}\int_\Omega v^2 + \frac{\alpha}{\gamma-1}\int_\Omega |\nabla e|^2 + \frac{\alpha}{\gamma-1}\int_\Omega e^2\\
&= \left[\alpha - \left(\frac{1}{2}+\frac{m^2}{2c^2}\right)\right]\int_\Omega w^2 + \left[\frac{\alpha}{c^2} - \left(1+\frac{m^2}{2c^2}\right)\right]\int_\Omega v^2 + \left[\frac{\alpha}{\gamma-1} - \frac{1}{2}\right]\int_\Omega e^2\\
&\geq 0.
\end{aligned}$$

Thus, $A+\alpha I$ is a monotone operator for any positive

$$\alpha > \max\left\{|m|, \left(\frac{1}{2}+\frac{m^2}{2c^2}\right), c^2\left(1+\frac{m^2}{2c^2}\right), \frac{\gamma-1}{2}\right\}.$$

■

Claim 2. *For a positive constant $\beta > |m|$, $R(A + \beta I) = H$*

Proof of Claim 2. Take any $F = \begin{pmatrix} f \\ g \\ h \end{pmatrix} \in H$. We want to find $U = \begin{pmatrix} w \\ v \\ e \end{pmatrix} \in D(A)$ such that $(A + \beta I)U = F$. That is,

$$\begin{cases} -v + \beta w = f, \\ -c^2\Delta w - m^2 w + c^2\Delta e + \beta v = g, \\ (\gamma - 1)v - \sigma\Delta e + \beta e = h. \end{cases}$$

Taking $v = \beta w - f$, we have the following system:

$$\begin{cases} -\Delta w + \left(\dfrac{\beta^2 - m^2}{c^2}\right) w + \Delta e = \dfrac{\beta f}{c^2} + \dfrac{g}{c^2}, \\ -\Delta e + \dfrac{\beta}{\sigma} e + \dfrac{\beta(\gamma - 1)}{\sigma} w = \dfrac{(\gamma - 1)}{\sigma} f + \dfrac{h}{\sigma}. \end{cases} \tag{3.3.14}$$

Recall that by Remark 1.7.1, we have for any $p \in L^2(\Omega)$ and any constant $k > 0$, $-\Delta\phi + k\phi = p$ has a unique solution $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$. We want to find constants such that we can express 3.3.14 in the form $-\Delta\phi + k\phi = p$, where ϕ should be a linear combination of w and e.

For simplicity, we rewrite (3.3.14) as

$$-\Delta w + \gamma_1 w + \Delta e = f_1, \tag{3.3.15}$$
$$-\Delta e + \gamma_2 e + \gamma_3 w = f_2, \tag{3.3.16}$$

where $\gamma_1 = \frac{\beta^2 - m^2}{c^2}$, $\gamma_2 = \frac{\beta}{\sigma}$, $\gamma_3 = \frac{\beta(\gamma-1)}{\sigma}$, $f_1 = \frac{\beta}{c^2} f + \frac{g}{c^2}$, $f_2 = \frac{\gamma-1}{\sigma} f + \frac{h}{\sigma}$. Now, take some constant a to be fixed later. Multiplying (3.3.15) by a and adding it to (3.3.16) gives us

$$(-a\Delta w + a\gamma_1 w + a\Delta e) + (-\Delta e + \gamma_2 e + \gamma_3 w) = af_1 + f_2$$
$$\Rightarrow -\Delta(aw + (1 - a)e) + [(a\gamma_1 + \gamma_3)w + \gamma_2 e] = af_1 + f_2. \tag{3.3.17}$$

Now suppose there exists a constant k such that $\phi = aw + (1 - a)e$ and $k\phi = (a\gamma_1 + \gamma_3)w + \gamma_2 e$. We will try to find out if such constants a, k exist. So we have

$$k(aw + (1 - a)e) = (a\gamma_1 + \gamma_3)w + \gamma_2 e.$$

Comparing the coefficients of w and e,

$$\frac{a\gamma_1 + \gamma_3}{a} = \frac{\gamma_2}{1-a} = k. \tag{3.3.18}$$

That is,

$$\begin{aligned}
&(1-a)(a\gamma_1 + \gamma_3) = a\gamma 2 \\
&\Rightarrow a\gamma_1 + \gamma_3 - a^2\gamma_1 - a\gamma_3 = a\gamma_2 \\
&\Rightarrow \gamma_1 a^2 + (\gamma_2 + \gamma_3 - \gamma_1)a - \gamma_3 = 0.
\end{aligned} \tag{3.3.19}$$

The discriminant D of the above quadratic polynomial is $D = (\gamma_2+\gamma_3-\gamma_1)^2+4\gamma_1\gamma_3$. Note that $\gamma_1 = \frac{\beta^2-m^2}{c^2} > 0$ for $\beta > |m|$, $\gamma_3 = \frac{\beta(\gamma-1)}{\sigma} > 0$ since $\gamma > 1, \sigma > 0$, $\gamma_2 = \frac{\beta}{\sigma} > 0$. Thus $D > 0$ and the equation (3.3.19) has two distinct non-zero, real solutions (non-zero since $\gamma_3 > 0$). Let us denote the two solutions as a_1 and a_2 where

$$\begin{aligned}
a_1 &= \frac{-(\gamma_2+\gamma_3-\gamma_1) + \sqrt{D}}{2\gamma_1}, \\
a_2 &= \frac{-(\gamma_2+\gamma_3-\gamma_1) - \sqrt{D}}{2\gamma_1}.
\end{aligned}$$

Clearly, $a_1 > 0$ since $\sqrt{(\gamma_2+\gamma_3-\gamma_1)^2 + 4\gamma_1\gamma_2} > |\gamma_2+\gamma_3-\gamma_1|$. Let us denote the values of k corresponding to a_1 and a_2 as:

$$\begin{aligned}
k_1 &= \frac{a_1\gamma_1 + \gamma_3}{a_1} = \frac{\gamma_2}{1-a_1}, \\
k_2 &= \frac{a_2\gamma_1 + \gamma_3}{a_2} = \frac{\gamma_2}{1-a_2}.
\end{aligned}$$

Then,

$$\begin{aligned}
k_1 &= \frac{a_1\gamma_1+\gamma_3}{a_1} > 0, && \text{(since } a_1, \gamma_1, \gamma_3 > 0) \\
k_2 &= \frac{\gamma_2}{1-a_2} = \frac{2\gamma_1\gamma_2}{\gamma_1+\gamma_2+\gamma_3+\sqrt{D}} > 0. && \text{(since } \gamma_1, \gamma_2, \gamma_3 > 0)
\end{aligned}$$

Thus indeed there exist positive constants k_1, k_2 corresponding to constants a_1 and a_2. Note that by Remark 1.7.2, the equations

$$\begin{aligned}
-\Delta\phi_1 + k_1\phi_1 &= a_1 f_1 + f_2, \\
-\Delta\phi_2 + k_2\phi_2 &= a_2 f_1 + f_2
\end{aligned}$$

have unique solutions $\phi_1, \phi_2 \in H^2(\Omega) \cap H_0^1(\Omega)$. But by (3.3.17), $a_1 w + (1-a_1)e$ and $a_2 w + (1-a_2)e$ also satisfy the above equations. Thus by uniqueness we have

$$\begin{aligned}
\phi_1 &= a_1 w + (1-a_1)e, \\
\phi_2 &= a_2 w + (1-a_2)e.
\end{aligned}$$

Then we get,

$$w = \frac{(1-a_2)\phi_1 - (1-a_1)\phi_2}{(1-a_2)a_1 - (1-a_1)a_2},$$
$$e = \frac{a_2\phi_1 - a_1\phi_2}{(1-a_1)a_2 - (1-a_2)a_1}.$$

Clearly, $w, e \in H^2(\Omega) \cap H_0^1(\Omega)$. Putting $v = \beta w - f$, we have found $U = \begin{pmatrix} w \\ v \\ e \end{pmatrix} \in D(A)$ such that $(A + \beta I)U = F$. That is, $R(A + \beta I) = H$. ■

Now, continuing with the proof of Theorem 3.3.1, if we choose

$$\alpha > \max\left\{|m|, \left(\frac{1}{2} + \frac{m^2}{2c^2}\right), c^2\left(1 + \frac{m^2}{2c^2}\right), \frac{\gamma - 1}{2}\right\}$$

and $\beta = \alpha + 1$, then $\beta > \alpha > |m|$ and by *Claim 1* and *Claim 2*, $A + \alpha I$ is maximal monotone. Denote $A_1 = A + \alpha I$. Consider the problem

$$\begin{cases} \dfrac{\mathrm{d}V}{\mathrm{d}t} + A_1 V = 0 \quad \text{on} \quad [0, \infty), \\ V(0) = U(0) \ \in D(A_1) = D(A). \end{cases} \tag{3.3.20}$$

Since A_1 is maximal monotone, by Hille-Yosida Theorem (Theorem 2.1.5) , (3.3.20) has a unique solution

$$V \in C^1([0, \infty); H) \cap C([0, \infty); D(A))$$

<u>*Remark 3.3.1.*</u> *Recall that in the proof of Lemma 2.1.2, we showed that if w satisfies*

$$\frac{\mathrm{d}w}{\mathrm{d}t} + A_\lambda w = 0 \tag{$\otimes$}$$

then $t \mapsto |w(t)|$ is non-increasing. We also showed that if w satisfies $\otimes$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}w}{\mathrm{d}t}\right) + A_\lambda\left(\frac{\mathrm{d}w}{\mathrm{d}t}\right) = 0$$

Then we also have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}w}{\mathrm{d}t} + \alpha w\right) + A_\lambda\left(\frac{\mathrm{d}w}{\mathrm{d}t} + \alpha w\right) = 0$$

and $t \mapsto \left|\frac{\mathrm{d}w}{\mathrm{d}t}(t) + \alpha w(t)\right|$ *is non-increasing. Using this and following the proof of Theorem 2.1.5, we would have the estimate*

$$\|-Au(t) + \alpha u(t)\| \leq \|-Au_0 + \alpha u_0\|.$$

From Remark 3.3.1, we have the following estimates for the solution V of (3.3.20), for $t \geq 0$

$$\|V(t)\| \leq \|U_0\| \tag{3.3.21}$$

and

$$\begin{aligned} &\|-A_1V(t) + \alpha V(t)\| \leq \|-A_1U_0 + \alpha U_0\| \\ \Rightarrow\ &\|-AV(t) - \alpha V(t) + \alpha V(t)\| \leq \|-AU_0 - \alpha U_0 + \alpha U_0\| \\ \Rightarrow\ &\|AV(t)\| \leq \|AU_0\|. \end{aligned} \tag{3.3.22}$$

Taking $U(t) = e^{\alpha t}V(t)$, we have

$$\begin{aligned} \frac{\mathrm{d}U}{\mathrm{d}t}(t) &= e^{\alpha t}\left[\frac{\mathrm{d}V}{\mathrm{d}t}(t) + \alpha V(t)\right] \\ &= e^{\alpha t}\left[-AV(t)\right] \qquad \text{(from (3.3.20))} \\ &= -A(e^{\alpha t}V(t)) = -AU(t). \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{\mathrm{d}U}{\mathrm{d}t}(t) + AU(t) = 0 \quad \text{on } [0,\infty), \\ &U(0) = V(0) = U_0. \end{aligned}$$

Therefore, $U(t) = e^{\alpha t}V(t)$ is the unique solution to the problem

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} + AU = 0 \quad \text{on } [0,\infty), \\ U(0) = U_0, \end{cases}$$

satisfying $U \in C^1([0,\infty); H) \cap C([0,\infty); D(A))$ from which (3.3.3) and (3.3.4) follow. Besides, from the estimates (3.3.21) and (3.3.22), we have

$$\begin{aligned} \|U(t)\|^2 &= e^{2\alpha t}\|V(t)\|^2 \leq e^{2\alpha t}\|U_0\|^2, \\ \|AU(t)\|^2 &= e^{2\alpha t}\|AV(t)\|^2 \leq e^{2\alpha t}\|AU_0\|^2, \end{aligned}$$

from which (3.3.5) and (3.3.6) follow. This concludes the proof of Theorem 3.3.1. ∎

Regularity of solution

Proof of Theorem 3.3.2. Recall that we define $D(A^k)$ inductively as follows

$$D(A^1) = D(A); D(A^k) = \left\{U \in D(A^{k-1}) | AU \in D(A^{k-1})\right\}.$$

We sill show by induction that

$$D(A^k) \subseteq D_k = \left\{ \begin{pmatrix} w \\ v \\ e \end{pmatrix} \middle| \begin{array}{lll} w \in H^{k+1}(\Omega), & \Delta^j w = 0 \text{ on } \Gamma & \forall j, 0 \le j \le \left[\frac{k}{2}\right] \\ v \in H^k(\Omega), & \Delta^j v = 0 \text{ on } \Gamma & \forall j, 0 \le j \le \left[\frac{k-1}{2}\right] \\ e \in H^{k+1}(\Omega), & \Delta^j e = 0 \text{ on } \Gamma & \forall j, 0 \le j \le \left[\frac{k}{2}\right] \end{array} \right\}.$$

For $k = 1$, we have

$$\begin{aligned} D(A) &= (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega)) \\ &= \left\{ \begin{pmatrix} w \\ v \\ e \end{pmatrix} \middle| \begin{array}{ll} w \in H^{k+1}(\Omega), & w = 0 \text{ on } \Gamma \\ v \in H^k(\Omega), & v = 0 \text{ on } \Gamma \\ e \in H^{k+1}(\Omega), & e = 0 \text{ on } \Gamma \end{array} \right\} = D_1. \end{aligned}$$

Then, $D(A) \subseteq D_1$. Now suppose $D(A^k) \subseteq D_k$. For $k + 1$, we have

$$\begin{aligned} &D(A^{k+1}) \\ &= \left\{U \in D(A^k) \subseteq D_k \mid AU \in D(A^k) \subseteq D_k\right\} \\ &\subseteq \left\{ \begin{pmatrix} w \\ v \\ e \end{pmatrix} \middle| \begin{array}{lll} w \in H^{k+1}(\Omega), & \Delta^j w = 0 \text{ on } \Gamma & \forall j, 0 \le j \le \left[\frac{k}{2}\right] \\ v \in H^k(\Omega), & \Delta^j v = 0 \text{ on } \Gamma & \forall j, 0 \le j \le \left[\frac{k-1}{2}\right] \\ e \in H^{k+1}(\Omega), & \Delta^j e = 0 \text{ on } \Gamma & \forall j, 0 \le j \le \left[\frac{k}{2}\right] \end{array} \right\} \\ &\bigcap \left\{ \begin{pmatrix} w \\ v \\ e \end{pmatrix} \middle| \begin{array}{l} -v \in H^{k+1}(\Omega), \ \Delta^j(-v) = 0 \text{ on } \Gamma, \forall j, 0 \le j \le \left[\frac{k}{2}\right] \\ -c^2\Delta w - m^2 w + c^2 \Delta e \in H^k(\Omega), \\ \Delta^j(-c^2\Delta w - m^2 w + c^2\Delta e) = 0 \text{ on } \Gamma, \forall j, 0 \le j \le \left[\frac{k-1}{2}\right] \\ (\gamma - 1)v - \sigma\Delta e \in H^{k+1}(\Omega), \\ \Delta^j((\gamma - 1)v - \sigma\Delta e) = 0 \text{ on } \Gamma, \forall j, 0 \le j \le \left[\frac{k}{2}\right] \end{array} \right\}. \end{aligned} \tag{3.3.23}$$

Note that we have $-v \in H^{k+1}(\Omega)$. This together with $(\gamma - 1)v - \sigma\Delta e \in H^{k+1}(\Omega)$ implies $\Delta e \in H^{k+1}(\Omega)$. Also we have $e \in H^{k+1}(\Omega)$. By *Remark 1.7.1*, $\exists$ a unique $f_e \in L^2(\Omega)$ such that $-\Delta e + e = f_e$. But then $f_e \in H^{k+1}(\Omega)$ and by Theorem 1.7.2, $e \in H^{k+3}(\Omega)$.

Again, note that we have $-c^2\Delta w - m^2 w + c^2\Delta e \in H^k(\Omega)$, $w \in H^{k+1}(\Omega)$ and $\Delta e \in H^{k+1}(\Omega)$. These together imply $\Delta w \in H^k(\Omega)$. And by a similar argument as above, we have $w \in H^{k+2}(\Omega)$.

From (3.3.23), we also get $\Delta^j e = \frac{\gamma-1}{\sigma}\Delta^{j-1} v = 0$ for $1 \le j \le [k/2] + 1$. This together with $\Delta^j e = 0$ on Γ for $0 \le j \le [k/2]$ gives us

$$\Delta^j e = 0 \text{ on } \Gamma \text{ for } 0 \le j \le \left[\frac{k+1}{2}\right].$$

Again, we have $\Delta^{j+1}w = \frac{m^2\Delta^j w - c^2\Delta^{j+1}e}{-c^2} = 0$ on Γ for $0 \leq j \leq \left[\frac{k-1}{2}\right]$. That is, $\Delta^j w = 0$ on Γ for $0 \leq j \leq \left[\frac{k-1}{2}\right] + 1 = \left[\frac{k+1}{2}\right]$. This together with $\Delta^j w = 0$ on Γ for $0 \leq j \leq [k/2]$ gives us that

$$\Delta^j w = 0 \text{ on } \Gamma \text{ for } 0 \leq j \leq \left[\frac{k}{2}\right] = \left[\frac{(k+1)-1}{2}\right].$$

And hence, we conclude that $D(A^{k+1}) \subseteq D_{k+1}$. Thus by induction, $D(A^k) \subseteq D_k$, $\forall$ integers $k \geq 1$. In fact, we have $D(A^k) = D_k$. In particular note that

$$D(A^k) \subseteq H^{k+1}(\Omega) \times H^k(\Omega) \times H^{k+1}(\Omega).$$

We make the following claim.

Claim 3. $D(A^k) \subseteq H^{k+1}(\Omega) \times H^k(\Omega) \times H^{k+1}(\Omega)$ *is a continuous injection* $\forall$ *integers* $k \geq 1$.

Proof of Claim 3. We will use induction on k. For $k = 1$, we have $D(A) \subset H^2(\Omega) \times H^1(\Omega) \times H^2(\Omega)$. Note that by *Remark 1.7.2*, we have

$$\begin{aligned} \|w\|_{H^2}^2 &\leq 2C_1^2\left(\|\Delta w\|_{L^2}^2 + \|w\|_{L^2}^2\right) \\ &\leq 2C_1^2\left(\|\Delta w\|_{L^2}^2 + \|w\|_{H^1}^2\right), \end{aligned} \tag{3.3.24}$$

$$\begin{aligned} \|e\|_{H^2}^2 &\leq 2C_2^2\left(\|\Delta e\|_{L^2}^2 + \|e\|_{L^2}^2\right) \\ &\leq 2C_2^2\left(\|\Delta e\|_{L^2}^2 + \|e\|_{H^1}^2\right), \end{aligned} \tag{3.3.25}$$

where C_1 and C_2 are constants depending only on Ω. Recall that in $D(A)$ we have the norm

$$\|U\|_{D(A)}^2 = \|U\|^2 + \|AU\|^2. \tag{3.3.26}$$

Note that for $U = \begin{pmatrix} w \\ v \\ e \end{pmatrix}$,

$$\|U\|^2 = \|w\|_{H^1}^2 + \frac{1}{c^2}\|v\|_{L^2}^2 + \frac{1}{\gamma-1}\|e\|_{H^1}^2, \tag{3.3.27}$$

$$\|AU\|^2 = \|-v\|_{H^1}^2 + \frac{1}{c^2}\left\|-c^2\Delta w - m^2 w + c^2\Delta e\right\|_{L^2}^2 + \frac{1}{\gamma-1}\|(\gamma-1)v - \sigma\Delta e\|_{H^1}^2. \tag{3.3.28}$$

Now if $\|U\|_{D(A)} \to 0$, then from (3.3.27) and (3.3.28), we have:

$$\|w\|_{H^1} \to 0, \tag{3.3.29}$$
$$\|e\|_{H^1} \to 0, \tag{3.3.30}$$
$$\|v\|_{H^1} \to 0, \tag{3.3.31}$$
$$\|(\gamma-1)v - \sigma\Delta e\|_{H^1} \to 0, \tag{3.3.32}$$
$$\left\|-c^2\Delta w - m^2 w + c^2\Delta e\right\|_{L^2} \to 0. \tag{3.3.33}$$

From (3.3.31) and (3.3.32), it follows that

$$\|\Delta e\|_{L^2} \to 0 \quad \text{as} \quad \|U\|_{D(A)} \to 0. \tag{3.3.34}$$

From (3.3.34), (3.3.33) and (3.3.29), we get

$$\|\Delta w\|_{L^2} \to 0 \quad \text{as} \quad \|U\|_{D(A)} \to 0. \tag{3.3.35}$$

Then from (3.3.35) and (3.3.24), we have

$$\|w\|_{H^2} \to 0 \quad \text{as} \quad \|U\|_{D(A)} \to 0. \tag{3.3.36}$$

Again, by (3.3.34) and (3.3.25), it follows

$$\|e\|_{H^2} \to 0 \quad \text{as} \quad \|U\|_{D(A)} \to 0. \tag{3.3.37}$$

Finally combining (3.3.37), (3.3.36) and (3.3.31), we have

$$\begin{aligned}\|U\|^2_{H^2\times H^1\times H^2} &= \|w\|^2_{H^2} + \|v\|^2_{H^1} + \|e\|^2_{H^2} \\ &\to 0 \ \text{ as } \ \|U\|_{D(A)} \to 0.\end{aligned}$$

That is, $D(A) \subseteq H^2(\Omega)\times H^1(\Omega)\times H^2(\Omega)$ is a continuous injection. Now, let's assume *Claim 3* holds for k. Note that

$$\begin{aligned}\|U\|^2_{D(A^{k+1})} &= \sum_{j=0}^{k+1} \left\|A^j U\right\|^2 \\ &= \|U\|^2_{D(A^k)} + \left\|A^{k+1}U\right\|^2 \qquad (3.3.38)\\ &= \|U\|^2 + \|AU\|^2_{D(A^k)}. \qquad (3.3.39)\end{aligned}$$

So from (3.3.38), we have

$$\begin{aligned}&\|U\|^2_{D(A^{k+1})} \to 0 \\ \Rightarrow\ &\|U\|^2_{D(A^k)} \to 0 \\ \Rightarrow\ &\|U\|^2_{H^{k+1}\times H^k\times H^{k+1}} \to 0 \qquad \text{(by induction hypothesis)} \\ \Rightarrow\ &\|w\|^2_{H^{k+1}} + \|v\|^2_{H^k} + \|e\|^2_{H^{k+1}} \to 0. \qquad (3.3.40)\end{aligned}$$

Now, from (3.3.39), we have

$$\begin{aligned}
&\|U\|^2_{D(A^{k+1})} \to 0 \\
\Rightarrow\ &\|AU\|^2_{D(A^k)} \to 0 \\
\Rightarrow\ &\|AU\|^2_{H^{k+1}\times H^k\times H^{k+1}} \to 0 && \text{(by induction hypothesis)} \\
\Rightarrow\ &\|-v\|^2_{H^{k+1}} + \left\|c^2\Delta w + m^2 w - c^2\Delta e\right\|^2_{H^k} + \|(\gamma-1)v - \sigma\Delta e\|^2_{H^{k+1}} \to 0. && (3.3.41)
\end{aligned}$$

From (3.3.40) and (3.3.41), we find

$$\|\Delta w\|_{H^k}, \|v\|_{H^{k+1}}, \|\Delta e\|_{H^{k+1}} \to 0 \ \text{ as } \ \|U\|_{D(A^{k+1})} \to 0. \tag{3.3.42}$$

Again, by *Remark 1.7.2*, we have

$$\begin{aligned}
\|w\|^2_{H^{k+2}} &\le 2C_1^2\left(\|\Delta w\|^2_{H^k} + \|w\|^2_{H^k}\right) \\
&\le 2C_1^2\left(\|\Delta w\|^2_{H^k} + \|w\|^2_{H^{k+1}}\right), && (3.3.43) \\
\|e\|^2_{H^{k+2}} &\le 2C_2^2\left(\|\Delta e\|^2_{H^k} + \|e\|^2_{H^k}\right) \\
&\le 2C_2^2\left(\|\Delta e\|^2_{H^k} + \|e\|^2_{H^{k+1}}\right). && (3.3.44)
\end{aligned}$$

So combining (3.3.44), (3.3.43),(3.3.42) and (3.3.40), we find

$$\|w\|^2_{H^{k+2}}, \|v\|^2_{H^{k+1}}, \|e\|^2_{H^{k+2}} \to 0 \ \text{ as } \ \|U\|_{D(A^{k+1})} \to 0.$$

Therefore,

$$\begin{aligned}
\|U\|^2_{H^{k+2}\times H^{k+1}\times H^{k+2}} &= \|w\|^2_{H^{k+2}} + \|v\|^2_{H^{k+1}} + \|e\|^2_{H^{k+2}} \\
&\to 0 \ \text{ as } \ \|U\|_{D(A^{k+1})} \to 0.
\end{aligned}$$

That is the injection $D(A^{k+1}) \subseteq H^{k+2}(\Omega)\times H^{k+1}(\Omega)\times H^{k+2}(\Omega)$ is continuous. Hence, by induction, *Claim 3* holds. ■

Now we continue with the proof of Theorem 3.3.2. First note that $D(A^k) = D(A_1^k)$ (where as before, $A_1 = A+\alpha I$) and it can be easily checked that $\|\cdot\|_{D(A^k)}$ and $\|\cdot\|_{D(A_1^k)}$ are equivalent norms in $D(A^k) = D(A_1^k)$.

Now, given $U_0 \in D(A^k)$, by Theorem 2.2.1, we have that the unique solution V of the problem

$$\begin{cases} \dfrac{\mathrm{d}V}{\mathrm{d}t} + A_1 V = 0 \ \text{ on } [0,\infty), \\ V(0) = U_0 \end{cases}$$

satisfies $V \in C^{k-j}([0,\infty); D(A_1^j)) \;\; \forall\, j = 0, 1, ..., k$. That is,

$$V \in C^{k-j}([0,\infty); D(A^j)) \;\; \forall\, j = 0, 1, ..., k.$$

Then the unique solution $U(t) = e^{\alpha t} V(t)$ of the problem

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} + AU = 0 \;\; \text{on } [0,\infty), \\ U(0) = U_0 \end{cases}$$

satisfies

$$U \in C^{k-j}([0,\infty); D(A^j)) \;\; \forall\, j = 0, 1, ..., k. \tag{3.3.45}$$

Now, since by assumption in Theorem 3.3.2, $U_0 \in D(A^k)$ for all $k \in \mathbb{N}$, (3.3.45) holds for all $k \in \mathbb{N}$.

Note that from Corollary 1.5.3, given any $l \in \mathbb{N}$, $\exists$ an integer $m > l$ such that $H^m(\Omega) \subset C^l(\overline{\Omega})$ is a continuous injection. Then clearly, so is $H^{m+1}(\Omega) \subset H^m(\Omega) \subset C^l(\overline{\Omega})$. Choose $k = l + m, j = m$. Then (3.3.46) reads

$$\begin{aligned} &U \in C^l\left([0,\infty); H^{m+1}(\Omega) \times H^m(\Omega) \times H^{m+1}(\Omega)\right) \\ \Rightarrow\ &U \in C^l([0,\infty); C^l(\overline{\Omega}) \times C^l(\overline{\Omega}) \times C^l(\overline{\Omega})). \end{aligned} \tag{3.3.46}$$

Since (3.3.46) holds for any $l \in \mathbb{N}$, it follows

$$U \in C^\infty([0,\infty); C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega})).$$

That is,

$$(w, e) \in C^\infty(\overline{\Omega} \times [0,\infty)) \times C^\infty(\overline{\Omega} \times [0,\infty)).$$

This concludes the proof of Theorem 3.3.2.

■

Bibliography

[1] H. Brézis, *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert*, North-Holland Pub. Co, Amsterdam, 1973.

[2] H. Brézis, *Functional Analysis, Sobolev Spaces and Partial Differential equations*, Springer, New York London, 2011.

[3] A. Carasso, *Coupled sound and heat flow and the method of least squares*, Math. Comp., 29 (1975), pp. 447–463.

[4] R. Dautray and J. L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Springer-Verlag, Berlin-New York, 1988.

[5] L. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, R.I, 2010.

[6] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin-New York, 2001.

[7] F. Harlow and A. Amsden, *Fluid Dynamics, LASL Monograph LA 4700*, Los Alamos Scientific Laboratories, Los Alamos, N. M., 1971.

[8] V. Hokkanen and G. Moroşanu, *Functional Methods in Differential equations*, Chapman & Hall/CRC, Boca Raton, 2002.

[9] T. Kato, *Perturbation Theory for Linear operators*, Springer-Verlag, Berlin, 1966.

[10] P. Lax, *Functional Analysis*, Wiley, New York, 2002.

[11] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications (3 volumes)*, Springer Berlin-Heidelberg, 1972.

[12] A. Matsubara and T. Yokota, *Applications of the Hille-Yosida theorem to the linearized equations of coupled sound and heat flow*, AIMS Mathematics, 1 (2016), pp. 165–177.

[13] G. Moroșanu, *Nonlinear Evolution Equations and Applications*, Editura Academiei D. Reidel Distributors for the U.S.A. and Canada, Kluwer Academic Publishers, Bucuresti, Romania Dordrecht Boston Norwell, MA, 1988.

[14] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, 1983.

[15] R. D. Richtmyer and K. W. Morton, *Difference Methods for Initial-value Problems*, Krieger Pub. Co, Malabar, Fla, 1994.

[16] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, New York, 1976.

[17] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1991.

[18] K. Yoshida, *Functional Analysis*, Springer, Berlin-New York, 1995.