# On the size of the largest $P$-free families

by

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# Contents

**Abstract**

We will study the problem of determining the maximum size of a family of subsets of $[n] = \{1, 2, \ldots, n\}$ not containing a given poset $P$ as a (weak) subposet, denoted $La(n, P)$. This problem is a generalization of the well-known Sperner's theorem. In 1945, Erdős obtained the exact value of $La(n, P)$ when $P$ is a *path poset*, generalizing Sperner's theorem. A more formal study of this problem was initiated by Katona and Tarján in 1983. Since then there have been numerous papers in this area and many open questions. One of the open questions was to obtain a good general bound on $La(n, P)$ for an arbitrary poset $P$. Open questions concerning the exact (or at least asymptotic) value of $La(n, P)$ for some specific posets $P$ are also of great interest. The most famous poset of which, is the *Diamond*. In this thesis, we answer some of these open questions. We obtain a general bound on $La(n, P)$ which is best possible upto a constant factor, improving the previous bounds due to Burcsi and Nagy and later Chen and Li. We also obtain the exact value of $La(n, P)$ for an infinite class of posets and introduce a new method for doing so.

The thesis consists of 3 chapters: In the first chapter we survey results about forbidden subposets and prove some well-known theorems.

In the second chapter we show $\mathrm{La}(n, P) \leq \frac{1}{2^{k-1}} \left(|P| + (3k-5)2^{k-2}(h(P)-1) - 1\right) \binom{n}{\lfloor n/2 \rfloor}$ for any fixed integer $k \geq 2$, improving the best known upper bound. By choosing $k$ appropriately, we obtain that $\mathrm{La}(n, P) = \mathcal{O}\left(h(P) \log_2 \left(\frac{|P|}{h(P)} + 2\right)\right) \binom{n}{\lfloor n/2 \rfloor}$ as a corollary, which we show is best possible for general $P$. We also give a different proof of this corollary by using bounds for generalized diamonds. We also show that the Lubell function of a family of subsets of $[n]$ not containing $P$ as an induced subposet is $\mathcal{O}(n^c)$ for every $c > \frac{1}{2}$. This is joint work with Dániel Grósz and Casey Tompkins.

In the third chapter, we introduce a method of decomposing the family of intervals along a cyclic permutation into chains to determine the exact size of the largest family of subsets of $[n]$ not containing one or more given posets as a subposet. De Bonis, Katona and Swanepoel determined the size of the largest butterfly-free family. We strengthen this result by showing that, for certain posets containing the butterfly poset as a subposet, the same bound holds. We also obtain the corresponding LYM-type inequalities. This is joint work with Casey Tompkins.

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# Chapter 1

# Introduction

**Definition 1.** *Let $P$ be a finite poset, and $\mathcal{F}$ be a family of subsets of $[n]$. We say that $P$ is contained in $\mathcal{F}$ as a* weak *subposet if and only if there is an injection $\alpha : P \to \mathcal{F}$ satisfying $x_1 <_p x_2 \Rightarrow \alpha(x_1) \subset \alpha(x_2)$ for all $x_1, x_2 \in P$. $\mathcal{F}$ is called $P$-free if $P$ is not contained in $\mathcal{F}$ as a weak subposet. We define the corresponding extremal function as*

$$La(n, P) := \max\{|\mathcal{F}| \mid \mathcal{F} \text{ is } P\text{-free}\}.$$

*We say that $P$ is an* induced subposet *of $Q$ if there exists an injection $\alpha : P \to \mathcal{F}$ satisfying $x_1 <_p x_2 \iff \alpha(x_1) \subset \alpha(x_2)$ for all $x_1, x_2 \in P$. $\mathcal{F}$ is called induced $P$-free if $P$ is not contained in $\mathcal{F}$ as an induced subposet. We define the corresponding extremal function as*

$$La^{\#}(n, P) := \max\{|\mathcal{F}| \mid \mathcal{F} \text{ is induced } P\text{-free}\}.$$

In this thesis, we mostly study the first extremal function. If we wish to forbid a pair of posets $P$ and $Q$, we simply write $\mathrm{La}(n, P, Q)$ and $\mathrm{La}^{\#}(n, P, Q)$ respectively. We denote the number of elements of a poset $P$ by $|P|$. The linearly ordered poset on $k$ elements, $a_1 < a_2 < \ldots < a_k$, is called a chain of length $k$, and is denoted by $P_k$. Using our notation Sperner's theorem can be stated as follows.

**Theorem 1** (Sperner [27])**.**

$$La(n, P_2) = \binom{n}{\lfloor n/2 \rfloor}.$$

*Proof.* Let $\mathcal{A}$ be a $P_2$-free family. Consider a maximal chain,

$$\mathcal{C} = \{\varnothing, \{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_3, x_4\}, \ldots, [n]\}$$

formed by adding sequentially the elements $x_1, x_2, \ldots, x_n$ in this order. Let us double count the number of pairs $(A, \mathcal{C})$ where $A \in \mathcal{A}$ and $A \in \mathcal{C}$. For a fixed $A \in \mathcal{A}$, the number of maximal chains containing $A$ is $|A|!\,(n - |A|)!$. So,

$$|\{(A, \mathcal{C}) \mid A \in \mathcal{A} \text{ and } A \in \mathcal{C}\}| = \sum_{A \in \mathcal{A}} |A|!\,(n - |A|)! \tag{1.1}$$

On the other hand, a fixed maximal chain $\mathcal{C}$ has at most 1 set from $\mathcal{A}$ for otherwise we will have $P_2$ as a subposet of $\mathcal{A}$, contradiction. Clearly there are $n!$ maximal chains. So,

$$|\{(A,\mathcal{C}) \mid A \in \mathcal{A} \text{ and } A \in \mathcal{C}\}| \leq n! \tag{1.2}$$

Combining (1.1) and (1.2) proves our theorem. □

Erdős extended Sperner's theorem to $P_k$-free families for all $k \geq 2$.

**Theorem 2** (Erdős [9])**.** *$La(n, P_k)$ is equal to the sum of the $k-1$ largest binomial coefficients of order $n$. This implies*

$$La(n, P_k) \leq (k-1)\binom{n}{\lfloor n/2 \rfloor}.$$

*Proof.* The same proof of Theorem 1 applies. The only difference is that the right-hand side of (1.2) would now be $(k-1)n!$ because a fixed maximal chain can contain at most $(k-1)$ sets from a $P_k$-free family. □

Notice that, since any poset $P$ is a weak subposet of a chain of length $|P|$, Theorem 2 implies

**Corollary 1.**

$$La(n, P) \leq (|P|-1)\binom{n}{\lfloor n/2 \rfloor} = O\left(\binom{n}{\lfloor n/2 \rfloor}\right).$$

For a variety of posets, $P$, the value of $\mathrm{La}(n, P)$ has been determined asymptotically. The first forbidden poset result was due to Katona and Tarján [16] in 1983. They considered the $V$ poset defined on $\{x, y, z\}$ with relations $x \leq y, z$. They proved

$$\left(1 + \frac{1}{n} + o\left(\frac{1}{n}\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor} \leq \mathrm{La}(n, V) \leq \left(1 + \frac{2}{n}\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

This result was later generalized by De Bonis and Katona [7] who obtained bounds for the $r$-fork poset, $V_r$ defined by the relations $x \leq y_1, y_2, \ldots, y_r$. Other posets for which the asymptotic value of $La(n, P)$ has been determined include crowns $O_{2k}$ (cycle of length $2k$ on two levels) except for $k \in \{3, 5\}$ [19], the $N$ poset [10] and recently the complete 3 level poset $K_{r,s,t}$ [26] among others.

Fewer exact results are known. Already, in their paper introducing the La function, Katona and Tarjan [16] proved that $\mathrm{La}(n, V, \Lambda) = \mathrm{La}^{\#}(n, V, \Lambda) = 2\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}$, where $V$ and $\Lambda$ are the 2-fork and 2-brush, respectively. Define the butterfly poset, $B$, by 4 elements $a, b, c, d$ with $a, b \leq c, d$. Of central importance to the thesis is a theorem of De Bonis, Katona and Swanepoel [8] showing that $\mathrm{La}(n, B) = \Sigma(n, 2)$. More recently, several other exact results have been obtained. Burcsi and Nagy [4] obtained the exact bound for multiple posets and introduced a method of creating further posets whose La function could be calculated exactly. Griggs, Li and Lu [11] determined exact results for the $k$-diamond, $D_k$ ($w \leq x_1, x_2, \ldots, x_k \leq z$), for an infinite set of values of $k$. They also obtained exact results for harp posets, $\mathcal{H}(l_1, l_2, \ldots, l_k)$, defined by $k$ chains of length $l_i$ between two fixed elements, in the case when the $l_i$ are all distinct.

One of the first general results is due to Bukh who determined the asymptotic value of $La(n, P)$ for all posets whose Hasse diagram is a tree. Let $h(P)$ denote the height (maximum length of a chain) of $P$.

**Theorem 3** (Bukh [3])**.** *If $T$ is a finite poset whose Hasse diagram is a tree of height $h(T) \geq 2$, then*

$$La(n, T) = (h(T) - 1)\binom{n}{\lfloor n/2 \rfloor}(1 + O(1/n)). \tag{1.3}$$

Using a general structure called *double chain* instead of chains for double counting, Burcsi and Nagy obtained a similar but weaker version of this theorem for general posets thereby improving Corollary 1. Since some of their proof ideas are useful later on, we give their proof below. Before we state and prove their theorem we introduce the notion of a double chain.

**Definition 2** (Double chain)**.** *Let $\emptyset = A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_n = [n]$ be a maximal chain (so $|A_i| = i$). The double chain associated to this chain is given by $\mathcal{D} = \{A_0, A_1, \ldots, A_n, M_1, M_2, \ldots, M_{n-1}\}$, where $M_i = A_{i-1} \cup \{A_{i+1} \setminus A_i\}$.*

**Theorem 4** (Burcsi, Nagy [4])**.** *For any poset $P$, when $n$ is sufficiently large, we have*

$$\mathrm{La}(n, P) \leq \left(\frac{|P| + h(P)}{2} - 1\right)\binom{n}{\lfloor n/2 \rfloor}. \tag{1.4}$$

*Proof.* Let $\mathcal{A}$ be a $P$-free family. First let us observe that since there are $n!$ maximal chains in total, there are a total of $n!$ double chains associated to these maximal chains. Let us double count the number of pairs $(A, \mathcal{D})$ such that $A \in \mathcal{A}$ and $A \in \mathcal{D}$. It can be easily seen that for a fixed $A \in \mathcal{A}$, the number of double chains containing $A$ is $2\,|A|\,!\,(n - |A|)!$. So,

$$|\{(A, \mathcal{D}) \mid A \in \mathcal{A} \text{ and } A \in \mathcal{D}\}| = \sum_{A \in \mathcal{A}} 2\,|A|\,!\,(n - |A|)! \tag{1.5}$$

Now let us fix a double chain $\mathcal{D}$. We claim that there are at most $|P| + h(P) - 2$ sets of our $P$-free family $\mathcal{A}$ in $\mathcal{D}$. Suppose by contradiction that $\mathcal{A} \cap \mathcal{D}$ is of size at least $|P| + h(P) - 1$. We will show that then $\mathcal{D}$ contains $P$ as a subposet. For convenience, let us define the notion of an *infinite double chain*. An infinite double chain $\mathcal{D}_\infty$ is an infinite poset on elements $L_i, M_i$, $i \in \mathbb{Z}$ with relations $L_i \subset L_j, L_i \subset M_j$ and $M_j \subset L_i$ for all $i < j$. Clearly any poset formed by the sets in $\mathcal{A} \cap \mathcal{D}$ is a subposet of $\mathcal{D}_\infty$. So it suffices to show that any family $\mathcal{H} \subset \mathcal{D}_\infty$ of size at least $|P| + h(P) - 1$ contains $P$ as a subposet. To this end, first let us arrange the elements of $\mathcal{D}_\infty$ in the following order: $\ldots, L_{-1}, M_{-1}, L_0, M_0, L_1, M_1, \ldots$. This fixes an order on the elements of $\mathcal{H}$ as well. Now let us first decompose $P$ into antichains $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{h(P)}$ where the elements in $\mathcal{A}_i$ are bigger than or unrelated to elements in $\mathcal{A}_j$ for any $i > j$ and then map the antichains $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{h(P)}$ into $\mathcal{H}$ one after another, in this order, in $h(P)$ steps as follows. First we map $\mathcal{A}_1$ to the first $|\mathcal{A}_1|$ elements of $\mathcal{H}$ (according to the order we just defined on $\mathcal{H}$). Notice that all remaining elements of $\mathcal{H}$, except at most one element, are greater in $\mathcal{D}_\infty$ than the $|\mathcal{A}_1|$ elements that we just mapped to. Now we map $\mathcal{A}_2$ into the first $|\mathcal{A}_2|$ of these elements. Proceeding this way, it is easy to see that at each step we left at most one element unmapped in $\mathcal{H}$. So after $h(P)-1$ steps, we have left at most $h(P)-1$ elements unmapped so that we have at least $|P|+h(P)-1-\left(\sum_{i=1}^{h(P)-1}|\mathcal{A}_i|+h(P)-1\right)=\left|\mathcal{A}_{h(P)}\right|$ elements still remaining in $\mathcal{H}$ into which $\mathcal{A}_{h(P)}$ can be mapped. This shows that $P$ is a subposet of the poset defined by elements of $\mathcal{H}$, as desired. Since there are $n!$ double chains, we have

$$|\{(A,\mathcal{D}) \mid A\in\mathcal{A} \text{ and } A\in\mathcal{D}\}| \leq (|P|+h(P)-2)n! \tag{1.6}$$

Now we combine (1.5) and (1.6) to complete the proof. □

This result was improved by Chen and Li [5]. The idea of their proof was to generalize the double chain to a more complicated structure.

**Theorem 5** (Chen, Li [5])**.** *For any poset $P$, when $n$ is sufficiently large, the inequality*

$$La(n,P) \leq \frac{1}{m+1}\left(|P|+\frac{1}{2}(m^2+3m-2)(h(P)-1)-1\right)\binom{n}{\lfloor n/2\rfloor} \tag{1.7}$$

*holds for any fixed $m \geq 1$.*

Putting $m=\left\lceil\sqrt{\frac{|P|}{h(P)}}\right\rceil$ in the above formula, they obtained

$$\mathrm{La}(n,P)=\mathcal{O}(|P|^{1/2}\,h(P)^{1/2})\binom{n}{\lfloor n/2\rfloor}. \tag{1.8}$$

# Chapter 2

# A general bound on the largest family of subsets avoiding a subposet

In this chapter, we improve Theorem 5, by showing that

**Theorem 6** (Grósz, Methuku, Tompkins [14]). *For any poset $P$, when $n$ is sufficiently large, the inequality*

$$\mathrm{La}(n,P) \le \frac{1}{2^{k-1}}\left(|P| + (3k-5)2^{k-2}(h(P)-1) - 1\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}$$

*holds for any fixed $k \ge 2$.*

Notice that putting $k=2$, we get Theorem 4 and Theorem 5 for $m=1$. Putting $k=3$, we get Theorem 5 for $m=3$. For $k>3$, our result strictly improves Theorem 5.

By choosing $k$ appropriately in our theorem, we obtain the following improvement of (1.8):

**Corollary 2.** *For every poset $P$ and sufficiently large $n$,*

$$\mathrm{La}(n,P) = \mathcal{O}\left(h(P)\log_2\left(\frac{|P|}{h(P)} + 2\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

The following proposition shows that this bound cannot be improved for general $P$.

**Proposition 1** (Grósz, Methuku, Tompkins [14]). *For $P = K_{a,a,\ldots,a}$, we have*

$$\mathrm{La}(n,P) \ge ((h(P)-2)\log_2 a)\binom{n}{\lfloor \frac{n}{2} \rfloor} = \left((h(P)-2)\log_2\left(\frac{|P|}{h(P)}\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

It is interesting to note that much less is known about the induced version. The only known general bound on $\mathrm{La}^{\#}(n,P)$ has a much weaker constant than for the non-induced problem due to its dependence on the constant term of the higher dimensional variant of the Marcus-Tardos theorem [22, 17].

**Theorem 7** (Methuku, Pálvölgyi [23]). *For every poset $P$, there is a constant $C$ such that the size of any family of subsets of $[n]$ that does not contain an induced copy of $P$ is at most $C\binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

Define the Lubell function of a family of subsets of $[n]$ as $l_n(\mathcal{A}) = \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}}$. The Lubell function is the sum of the proportion of sets selected of each size; clearly $l_n(\mathcal{A}) \geq \frac{|A|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$. Define $\lambda_n^{\#}(P)$ as the maximum value of $l_n(\mathcal{A})$ over all induced $P$-free families $\mathcal{A} \subset 2^{[n]}$. While $\frac{\mathrm{La}^{\#}(n,P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$ is known to have a constant bound for every $P$, it is not currently known if $\lambda_n^{\#}(P)$ also has a constant bound for every $P$. We prove the following result about $\lambda_n^{\#}(P)$.

**Theorem 8** (Grósz, Methuku, Tompkins [14]). *For every poset $P$ and every $c > \frac{1}{2}$,*

$$\lambda_n^{\#}(P) = \mathcal{O}(n^c).$$

This chapter is organized as follows: In the first section we define our more general chain structure called an *interval chain* and give a proof of Theorem 6 and Corollary 2 using it. In the second section we give another proof of Corollary 2, with a better constant, using an embedding of arbitrary posets into a product of generalized diamonds. We also give a proof of Proposition 1. In the last section we use the interval chain technique to prove Theorem 8.

## 2.1 Interval chains and the proof of Theorem 6

We begin by proving some lemmas which allow us to extend Lubell's argument to more general structures. Let $\pi \in S_n$ be a permutation and $A \subset [n]$ be a set, then $A^\pi$ denotes the set $\{\pi(a) : a \in A\}$. Moreover, for a collection of sets $\mathcal{H} \subset 2^{[n]}$ we define $\mathcal{H}^\pi$ to be the collection $\{A^\pi : A \in \mathcal{H}\}$.

**Lemma 1.** *Let $\mathcal{H} \subset 2^{[n]}$ be a collection of sets and $A \subset [n]$ be any set. Let $N_i = N_i(\mathcal{H})$ be the number of sets in $\mathcal{H}$ of cardinality $i$. The number of permutations $\pi$ such that $A \in \mathcal{H}^\pi$ is $N_{|A|}\,|A|\,!\,(n - |A|)!$.*

*Proof.* Let $S_1, \ldots, S_{N_{|A|}}$ be the collection of sets in $\mathcal{H}$ of size $|A|$. The number of permutations $\pi$ such that $S_i$ is mapped to $A$ is $|A|\,!\,(n - |A|)!$, since we can map the elements of $S_i$ to $A$ arbitrarily and the elements of $[n] \setminus S_i$ to $[n] \setminus A$ arbitrarily. Moreover, no permutation $\pi$ maps two sets, $S_i, S_j$, to $A$, for then $S_i^\pi = S_j^\pi$, that is $\{\pi(s) : s \in S_i\} = \{\pi(s) : s \in S_j\}$ and so $S_i = S_j$, a contradiction. Since there are $N_{|A|}$ sets in $\mathcal{H}$ of size $|A|$, and we have shown that the set of permutations mapping each of them to $A$ is disjoint. It follows that the number of permutations $\pi$ such that $A \in \mathcal{H}^\pi$ is $N_{|A|}\,|A|\,!\,(n - |A|)!$. □

For a collection $\mathcal{H} \subset 2^{[n]}$ and a poset, $P$, let $\alpha(\mathcal{H}, P)$ denote the size of the largest subcollection of $\mathcal{H}$ containing no $P$. Observe that $\alpha(\mathcal{H}, P) = \alpha(\mathcal{H}^\pi, P)$ for all $\pi \in S_n$ since containment relations are unchanged by permutations of $[n]$.

**Lemma 2.** *Let $\mathcal{A}$ be a $P$-free family in $2^{[n]}$ and $\mathcal{H}$ be a fixed collection. We have*

$$\sum_{A\in\mathcal{A}} \frac{N_{|A|}}{\binom{n}{|A|}} \le \alpha(\mathcal{H}, P).$$

*In particular, if all of the $N_i$ are equal to the same number $N$, we have*

$$\sum_{A\in\mathcal{A}} \frac{1}{\binom{n}{|A|}} \le \frac{\alpha(\mathcal{H}, P)}{N}.$$

*Proof.* We will double count pairs $(A, \pi)$ where $A \in \mathcal{H}^\pi$. First fix a set $A$, then Lemma 1 shows there are $N_{|A|}\,|A|\,!\,(n-|A|)!$ permutations for which $A \in \mathcal{H}^\pi$. Now fix a permutation $\pi \in S_n$. By the definition of $\alpha(\mathcal{H}, P)$ we have $|\mathcal{A} \cap \mathcal{H}^\pi| \le \alpha(\mathcal{H}, P)$. Since there are $n!$ permutations, it follows that the number of pairs $(A, \pi)$ is at most $\alpha(\mathcal{H}, P)n!$. Thus, we have

$$\sum_{A\in\mathcal{A}} N_{|A|}\,|A|\,!\,(n-|A|)! \le \alpha(\mathcal{H}, P)n!\,,$$

and rearranging yields the result. □

We introduce a structure $\mathcal{H} \subset 2^{[n]}$ which we call a *$k$-interval chain*. Define the interval $[A, B]$ to be the set $\{C : A \subseteq C \subseteq B\}$. Fix a maximal chain $\mathcal{C} = \{A_0 = \emptyset, A_1, \ldots, A_{n-1}, A_n = [n]\}$ where $A_i \subset A_{i+1}$ for $0 \le i \le n-1$. From $\mathcal{C}$ we define the $k$-interval chain $\mathcal{C}_k$ as

$$\mathcal{C}_k = \bigcup_{i=0}^{n-k} [A_i, A_{i+k}].$$

See Figure 2.1 for an example of an interval chain. We begin by deriving some properties of interval chains. In the rest of the paper we shall work with the $k$-interval chain $\mathcal{C}_k^0$ defined by $A_i = [i]$; other $k$-interval chains are related to it by permutation. It is easy to see that the indicator vectors of the sets in $\mathcal{C}_k^0$ consist of an initial segment of 1's, then $k$ arbitrary bits, followed by 0's. We call the number of 1's in a 0–1 vector the weight of the vector (which is the size of the corresponding set).

We will now prove a sequence of lemmas that we use to bound the number of sets in a $P$-free subfamily of a $k$-interval chain. We call two sets related if one of them contains the other. The idea, following Burcsi, Nagy [4] and Chen, Li [5], is to partition $P$ into $h(P)$ antichains and embed the antichains into a given subcollection of $\mathcal{C}_k^0$, one by one, in such a way that every set in one antichain is related to every set in the next antichain. To this end, we ignore those sets in $\mathcal{C}_k^0$ which may be unrelated to some previously embedded set. The key lemma, Lemma 4, gives an upper bound to how many sets we must ignore.

For convenience, from now on we identify sets and their indicator vectors.

**Lemma 3.** *For $k \le m \le n-k$, the number of sets of size $m$ in a $k$-interval chain is $2^{k-1}$. The number of such sets which have at least $j$ 0's before the last 1 is $\sum_{h=j}^{k-1}\binom{k-1}{h}$.*

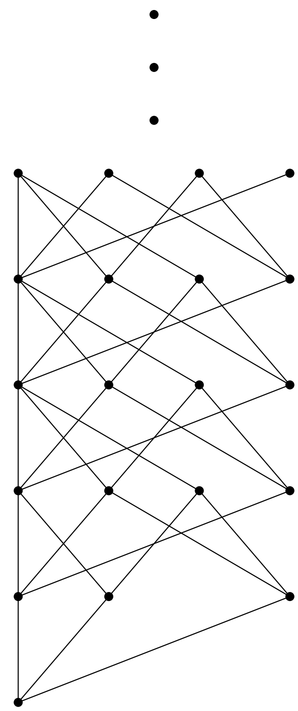

Figure 2.1: 3-interval chain

*Proof.* We give a bijection $\varphi$ between 0–1 vectors of length $k-1$ and sets of size $m$ in $\mathcal{C}_k^0$. Let $u$ be a 0–1 vector of length $k-1$, and let $w$ be the weight of $u$. Let $\varphi(u) = \overbrace{111\ldots1}^{m-w-1}\overbrace{u}^{k-1}1\overbrace{0000000\ldots0}^{n-m-k+w+1}$. A set of size $m$ in $\mathcal{C}_k^0$ is assigned to $u$ if and only if in its indicator vector the last $k-1$ bits leading up to (but not including) the last 1 coincide with $u$. We show $\varphi$ is injective and surjective. If $\varphi(u) = \varphi(v)$, then both $u$ and $v$ consist of the $k-1$ bits preceding the final 1 so $u = v$, and it follows $\varphi$ is injective. Now, take an arbitrary weight $m$ vector, $w$, corresponding to a set in $\mathcal{C}_k^0$. Find the last 1 occurring in $w$ and let $u$ be the vector of length $k-1$ immediately preceding it (such a vector exists since $m \geq k$). Then $\varphi(u) = w$, and we have that $\varphi$ is surjective.

There are $2^{k-1}$ vectors $u$ of length $k-1$. Among such vectors, $\sum_{h=j}^{k-1}\binom{k-1}{h}$ of them have at least $j$ 0's, and precisely these vectors are the ones mapped to vectors with at least $j$ 0's before the last 1. The condition $k \leq m \leq n-k$ guarantees that both $m-w-1$ and $m+k-w+1$ are between 0 and $n$. $\square$

**Lemma 4.** *For $3k-3 \leq m \leq n-k+1$, the number of sets in a $k$-interval chain which have size at most $m-1$, and which are unrelated to some other set in the $k$-interval chain of size at least $m$, is $(3k-5)2^{k-2}$.*

*Proof.* We will show that the sets in the $k$-interval chain $\mathcal{C}_k^0$, which are unrelated to at least one set of size $m$ or greater in $\mathcal{C}_k^0$ are: all indicator vectors in $\mathcal{C}_k^0$ of weight between $m-1$ and $m-(k-2)$ inclusive; plus, among indicator vectors with weight $m-i$ with $k-1 \leq i \leq 2k-3$, those which have at least $i-k+2$ 0's before the last 1. Let's denote the collection of these vectors by $\mathcal{S}$. Then, by Lemma 3, we can calculate the number $|\mathcal{S}|$ of such vectors:

$$(k-2)2^{k-1} + \sum_{i=k-1}^{2k-3} \sum_{h=i-k+2}^{k-1} \binom{k-1}{h} = (k-2)2^{k-1} + \sum_{j=1}^{k-1} \sum_{h=j}^{k-1} \binom{k-1}{h} =$$

$$= (k-2)2^{k-1} + \sum_{h=1}^{k-1} h\binom{k-1}{h} = (k-2)2^{k-1} + (k-1)2^{k-2} = (3k-5)2^{k-2}.$$

First we show that if $v \in \mathcal{S}$, there is a vector of weight $m$ in $\mathcal{C}_k^0$ which is unrelated to it. Let $m-i$ be the weight of $v$. We need to change at least one 1 to 0 (i.e., remove some elements), and change $i$ more 0's to 1's than we just removed (that is, add $i$ more elements than we just removed).

Assume that the last 1 in $v$ is at index $l$, so the first $l-k$ elements in $v$ are 1's. Also assume that there are $j$ 0's in $v$ with an index less than $l$. We can change $v_l$, the $l^{th}$ entry of $v$, from 1 to 0, and change the first $i+1$ 0's in $v$ to 1's because $i+1 \leq j+k-1$. We obtain either a vector with at least $l-k+2$ initial 1's, and 0's from an index $\leq l$; or a vector with $l-1$ initial 1's, and 0's from an index $\leq l+k-1$ (see the figure below). Either way the difference between the index of the last 1 and the first 0 is at most $k-1$, so the obtained vector is in $\mathcal{C}_k^0$.

$$\overbrace{111111111}^{\text{initial segment}}\overbrace{00010}^{\leq k-1}1\overbrace{00000}^{k-1}000 \qquad\qquad \overbrace{111111111}^{\text{initial segment}}\overbrace{00010}^{\leq k-1}1\overbrace{00000}^{k-1}000$$

$$\downarrow \qquad\qquad \text{or} \qquad\qquad \downarrow$$

$$\overbrace{111111111}\overbrace{11010}^{\leq k-1}0\overbrace{00000}^{k-1}000 \qquad\qquad \overbrace{111111111}\overbrace{11111}^{\leq k-1}0\overbrace{11000}^{k-1}000$$

Conversely, we prove that if $v$ (which is of weight at most $m-i$, $i \geq 1$) is not in $\mathcal{S}$, then it is related to all vectors of weight at least $m$ in $\mathcal{C}_k^0$. Assume by contradiction that it is unrelated to a vector $q$ in $\mathcal{C}_k^0$, of weight at least $m$.

Consider the transformation of $v$ into $q$ by changing some 1's to 0's and some 0's to 1's. Let $l'$ be the index of the first 1 that we change to 0. Then $l' \leq l$ (in the transformation given above, it was $l$, the index of the last 1). We can only change those bits from 0's to 1's which are before $l'$ (at most $j$), or those which are between $l'+1$ and $l'+k-1$ (at most $k-1$); this is because the new vector will have a 0 at index $l'$ and so it cannot have 1's after index $l'+k-1$ if it is in $\mathcal{C}_k^0$. So if $i+1 > j+k-1$, there are not enough 0's which could be changed to 1's, so we cannot obtain a vector of weight $m$ or greater, which is in $\mathcal{C}_k^0$ and is unrelated to it. □

*Observation* 1. The sets in $\mathcal{C}_k^0$ which are related to every set of size at least $m+1$ in $\mathcal{C}_k^0$, but unrelated to at least one set of size $m$ in $\mathcal{C}_k^0$ are those which have size $m-i$ with $k-2 \leq i \leq 2k-3$, and in whose indicator vector the number of 0's before the last 1 is exactly $i-k+2$. The only way we can obtain an indicator vector of weight $m$ corresponding to such a set in $\mathcal{C}_k^0$ is by removing the last 1, and changing all 0's before the last 1, plus the next $k-1$ after it, to 1's. Thus, there is only one set of size $m$ in $\mathcal{C}_k^0$ which is unrelated to these sets: the one with an indicator vector $\overbrace{111\ldots1}^{m-k+1}0\overbrace{11\ldots1}^{k-1}\overbrace{000\ldots0}^{n-m-1}$.

**Lemma 5.** *For any poset $P$ of size $|P|$ and height $h$, we have*

$$\alpha(\mathcal{C}_k, P) \leq |P| + (h-1)(3k-5)2^{k-2} - 1.$$

*Proof.* We show that if $\mathcal{H} \subseteq \mathcal{C}_k^0$ with $|\mathcal{H}| \geq |P| + (h-1)(3k-5)2^{k-2}$, then $\mathcal{H}$ contains $P$ as a subposet. We may notice that a $k$-interval chain on $[n]$ is a subposet of the levels $3k-3$ to $n'-k+1$ of a $k$-interval chain on the larger base set $[n']$ where $(n'-k+1)-(3k-3) = n$ (i.e., $n' = n+4k-4$), with the injection $2^{[n]} \ni A \mapsto \{1, 2, \ldots, 3k-3\} \cup \{a+3k-3 : a \in A\} \in 2^{[n']}$. So we can assume that the elements of $P$ are embedded from levels $3k-3$ to $n-k+1$ of the interval chain.

We define an order on $\mathcal{H}$: bigger sets come first; within sets of a given size $m$, the order is arbitrary, except if the set with the indicator vector $\overbrace{111\ldots1}^{m-k+1}0\overbrace{11\ldots1}^{k-1}\overbrace{000\ldots0}^{n-m-1}$ is present in $\mathcal{H}$, it must come last among the sets of size $m$.

Mirsky's theorem [25] states that the height of any poset equals the minimum number of antichains into which it can be partitioned. We decompose $P$ into antichains $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_h$, where the elements in $\mathcal{A}_i$ are bigger than or unrelated to elements in $\mathcal{A}_j$ for any $i > j$ and then map the antichains $\mathcal{A}_h, \mathcal{A}_{h-1}, \ldots, \mathcal{A}_1$ into $\mathcal{H}$ one after another, in this order, in $h$ steps as follows. First, we map the elements of $\mathcal{A}_h$ to the first $|\mathcal{A}_h|$ sets of $\mathcal{H}$ in the order just described. The family of these elements of $\mathcal{H}$ is denoted $\mathcal{H}_h$. We then remove all sets in $\mathcal{H}$ which are not proper subsets of every set in $\mathcal{H}_h$. The family of these removed sets is denoted $\mathcal{I}_h$; in other words, $\mathcal{I}_h$ is the family of sets in $\mathcal{H}$ which are not properly contained in at least one set of $\mathcal{H}_h$. (Notice that $\mathcal{H}_h \subseteq \mathcal{I}_h$.) Now we map $\mathcal{A}_{h-1}$ to the first $|\mathcal{A}_{h-1}|$ sets of $\mathcal{H} \setminus \mathcal{I}_h$, denoted $\mathcal{H}_{h-1}$. We proceed similarly: we denote the family of the sets in $\mathcal{H}$ which are not properly contained in every set of $\mathcal{H}_h \cup \ldots \cup \mathcal{H}_i$ with $\mathcal{I}_i$, and map $\mathcal{A}_{i-1}$ to the collection of first $|\mathcal{A}_{i-1}|$ sets of $\mathcal{H} \setminus \mathcal{I}_i$, denoted $\mathcal{H}_{i-1}$. By this process, each set in $\mathcal{H}_i$ contains all the sets in $\mathcal{H}_j$ for $i > j$.

We have to show that the process finishes before $\mathcal{H}$ is exhausted, that is,

$$\left|\bigcup_{i=1}^{h} \mathcal{H}_i \cup \bigcup_{i=2}^{h} \mathcal{I}_i\right| \leq |P| + (h-1)(3k-5)2^{k-2}. \tag{2.1}$$

For this purpose, we show that for each $i \in \{h, h-1, \ldots, 2\}$, the number of new sets that are removed at this step, besides $\mathcal{H}_i$: $|\mathcal{I}_i \setminus (\mathcal{H}_i \cup \mathcal{I}_{i+1})|$ is at most $(3k-5)2^{k-2}$ (where we consider $\mathcal{I}_{h+1} = \emptyset$). Since $\left|\bigcup_{i=1}^{h} \mathcal{H}_i\right| = |P|$ and there are $h(P)-1$ steps in which sets are removed, we will have our desired inequality (2.1). Let $A$ be the last set in $\mathcal{H}_i$ in the order we defined on $\mathcal{H}$, and $m = |A|$. Every set which comes before $A$ is either in $\mathcal{H}_i$ or $\mathcal{I}_{i+1}$. If $A = \overbrace{111\ldots1}^{m-k+1}0\overbrace{11\ldots1}^{k-1}\overbrace{000\ldots0}^{n-m-1}$, then $\mathcal{I}_i \setminus (\mathcal{H}_i \cup \mathcal{I}_{i+1})$ is a subcollection of all sets in $\mathcal{C}_k^0$ whose size is smaller than $m$, but which are unrelated to at least one set in $\mathcal{C}_k^0$ of size $m$ or more. By Lemma 4, the number of such sets is $(3k-5)2^{k-2}$. If $A \neq \overbrace{111\ldots1}^{m-k+1}0\overbrace{11\ldots1}^{k-1}\overbrace{000\ldots0}^{n-m-1}$, then, by Observation 1, the sets in $\mathcal{C}_k^0$ whose size is smaller than $m$, and which are unrelated to $A$ or some other set in $\mathcal{H}$ which is smaller than $A$ in our order, are also unrelated to some set in $\mathcal{C}_k^0$ of size $m+1$ or more. Thus the sets in $\mathcal{I}_i \setminus (\mathcal{H}_i \cup \mathcal{I}_{i+1})$ are some sets in $\mathcal{C}_k^0$ of size $m$ and some sets whose size is smaller than $m$ but unrelated to at least one set in $\mathcal{C}_k^0$ of size $m+1$ or more. Again, the number of such sets is at most $(3k-5)2^{k-2}$. □

Now we are ready to prove our main result, Theorem 6.

*Proof of Theorem 6.* Let $\mathcal{A}$ be a $P$-free family over $[n]$. Let $N_{|A|}$ denote the number of sets of size $|A|$ from the $k$-Interval chain.

$$2^{k-1}|\mathcal{A}| = \sum_{\substack{A\in\mathcal{A}\\ |A|<k \text{ or } |A|>n-k}} 2^{k-1} + \sum_{\substack{A\in\mathcal{A}\\ k\leq|A|\leq n-k}} 2^{k-1}$$
$$\leq \sum_{\substack{A\in\mathcal{A}\\ |A|<k \text{ or } |A|>n-k}} \frac{N_{|A|}\binom{n}{\lfloor\frac{n}{2}\rfloor}}{\binom{n}{|A|}} + \sum_{\substack{A\in\mathcal{A}\\ k\leq|A|\leq n-k}} \frac{2^{k-1}\cdot\binom{n}{\lfloor\frac{n}{2}\rfloor}}{\binom{n}{|A|}} \leq \alpha(\mathcal{C}_k, P)\binom{n}{\lfloor\frac{n}{2}\rfloor}.$$

If $|A| < k$ or $|A| > n-k$, we have $2^{k-1} \leq \frac{\binom{n}{\lfloor\frac{n}{2}\rfloor}}{\binom{n}{|A|}}$ when $n$ is sufficiently large and so the first inequality holds. If $k \leq |A| \leq n-k$, by Lemma 3, we have $2^{k-1} = N_{|A|}$ and so the second inequality holds due to Lemma 2. Now we use Lemma 5 to upper bound $\alpha(\mathcal{C}_k, P)$, from which the theorem follows. □

We now obtain Corollary 2 using the above theorem.

*First proof of Corollary 2.* Let $\mathcal{A}$ be a $P$-free family, and let $h$ be the height of $P$. Define $k = \left\lceil \log_2\left(\frac{|P|}{h}\right)\right\rceil = \log_2\left(\frac{|P|}{h}\right) + x = \log_2\left(\frac{|P|y}{h}\right)$. Let us substitute this $k$ into Theorem 6 (where $0 \leq x < 1$ and $1 \leq y < 2$). If $k \geq 2$, we get

$$\frac{|\mathcal{A}|}{\binom{n}{\lfloor\frac{n}{2}\rfloor}} \leq \frac{1}{2^{k-1}}\left(|P| + (h-1)(3k-5)2^{k-2} - 1\right)\binom{n}{\lfloor\frac{n}{2}\rfloor} < \frac{3\cdot 2^{k-2}kh + |P|}{2^{k-1}} =$$
$$= \frac{\frac{3}{4}y|P|\left(\log_2\left(\frac{|P|}{h}\right) + x\right) + |P|}{\frac{y|P|}{2h}} < \frac{3}{2}\log_2\left(\frac{|P|}{h}\right)h + 3.5h.$$

If $k \leq 1$, we have $|P| \leq 2h$. Double counting with just the chain gives a bound of $|P|\binom{n}{\lfloor\frac{n}{2}\rfloor}$ (see Erdős [9]), so the corollary still holds. So we have,

$$\mathrm{La}(n,P) < \left(\frac{3}{2}\log_2\left(\frac{|P|}{h}\right)h + 3.5h\right)\binom{n}{\lfloor\frac{n}{2}\rfloor}.$$ □

## 2.2 A different proof of Corollary 2 using generalized diamonds

We begin by recalling some results from the papers of Griggs and Li [12] and Griggs, Li and Lu [11].

**Definition 3** (Product of posets). *If a poset $P$ has a unique maximal element and a poset $Q$ has a unique minimal element, then their product $P \otimes Q$ is defined as the poset formed by identifying the maximal element of $P$ with the minimal element of $Q$.*

**Lemma 6** (Griggs, Li [12]). $\mathrm{La}(n, P \otimes Q) \leq \mathrm{La}(n, P) + \mathrm{La}(n, Q)$.

*Proof.* Let $\mathcal{F}$ be a maximal $P \otimes Q$-free family. Define $\mathcal{F}_1 = \{S \in \mathcal{F} \mid \mathcal{F} \cap [S, [n]] \text{ contains } Q\}$ and let $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$.

We claim that $\mathcal{F}_1$ is $P$-free. Suppose not. Then there is a set $M_1 \in \mathcal{F}_1$ which represents the maximal element of $P$, and, by definition, $\mathcal{F} \cap [M_1, [n]]$ contains $Q$. Also notice that, since $M_1$ represents the maximal element of $P$, there are no elements in $[M_1, [n]] \setminus \{M_1\}$ that are part of the representation of $P$. This implies that $\mathcal{F}$ contains $P \otimes Q$, a contradiction. It is easy to see that $\mathcal{F}_2$ is $Q$-free, for otherwise, the element $M_2$, that represents the minimal element of $Q$ satisfies: $\mathcal{F} \cap [M_2, [n]]$ contains $Q$, contradicting the definition of $\mathcal{F}_2$. So we have $|\mathcal{F}| = \mathrm{La}(n, P \otimes Q) = |\mathcal{F}_1| + |\mathcal{F}_2| \leq \mathrm{La}(n, P) + \mathrm{La}(n, Q)$, as desired. □

We shall write $h$ in place of $h(P)$ for convenience. Let $D_k$ be the poset on $k+2$ elements with relations $b < c_1, c_2, \ldots, c_k < d$. Let $K_{a_1,\ldots,a_h}$ be the complete $h$-level poset where the sizes of levels are $a_1, a_2, \ldots, a_h$: the poset in which every element is smaller than every element on every higher level.

By using a partition method on chains, Griggs, Li and Lu proved

**Theorem 9** (Griggs, Li, Lu [11]). *Let $k \geq 2$. Then,*

$$\mathrm{La}(n, D_k) \leq (\log_2(k+2) + 2)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

By Mirsky's decomposition [25], $P$ can be viewed as a union of $h$ antichains: $\mathcal{A}_i$, $1 \leq i \leq h$. Let $|\mathcal{A}_i| = a_i$. Then, it is easy to see that the following lemma holds.

**Lemma 7.** *$P$ is a subposet of $K_{a_1,\ldots,a_h}$, which in turn, is a subposet of $D_{a_1} \otimes D_{a_2} \otimes \ldots \otimes D_{h-1} \otimes D_{a_h}$.*

Now we are ready to prove Corollary 2 with better constants.

*Second proof of Corollary 2.* By Lemma 7, we have

$$\mathrm{La}(n, P) \leq \mathrm{La}(n, K_{a_1,\ldots,a_h}) \leq \mathrm{La}(n, D_{a_1} \otimes D_{a_2} \otimes \ldots \otimes D_{a_{h-1}} \otimes D_{a_h}).$$

By Lemma 6 and Theorem 9, we have

$$\mathrm{La}(n, D_{a_1} \otimes D_{a_2} \otimes \ldots \otimes D_{a_{h-1}} \otimes D_{a_h}) \leq \sum_{i=1}^{h} (\log_2(a_i + 2) + 2) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Bounding the sum on the right-hand side, by Jensen's inequality we have

$$\sum_{i=1}^{h} (\log_2(a_i + 2) + 2) \leq h \cdot \log_2 \left( \frac{|P|}{h} + 2 \right) + 2h.$$

This implies our desired result

$$\mathrm{La}(n, P) \leq \left( h \cdot \log_2 \left( \frac{|P|}{h} + 2 \right) + 2h \right) \binom{n}{\lfloor \frac{n}{2} \rfloor}. \qquad \square$$

Finally, we will prove Proposition 1, a matching lower bound for Corollary 2.

*Proof of Proposition 1.* We show that the height of any poset corresponding to a family of sets which realizes $K_{a,a,\ldots,a}$ is at least $(h-2)\log_2 a + 1$. This implies that if $\mathcal{A}$ is the middle $(h-2)\log_2 a$ levels of $2^{[n]}$, it does not contain $P$ as a subposet.

Let us denote the levels of $P = K_{a,a,\ldots,a}$ by $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_h$, and let $\mathcal{H}$ be a set family into which $P$ is embedded. For every $1 \leq i \leq h-1$, let $U_i$ be the union of the sets corresponding to the elements of $\mathcal{P}_i$ by the embedding. Then, the structure of $P$ implies that every element of $\mathcal{P}_{i+1}$ is mapped to sets containing $U_i$. If $|U_{i+1} \setminus U_i| = k$, there are $2^k$ sets in total containing $U_i$ and contained in $U_{i+1}$. Thus, we have $|U_{i+1}| - |U_i| \geq \log_2 a$ (this idea comes from Theorem 2.5 in [11]). So $|U_{h-1}| - |U_1| \geq (h-2)\log_2 a$. $\mathcal{P}_1$ is mapped to sets of size at most $|U_1|$, and $\mathcal{P}_h$ is mapped to sets of size at least $|U_{h-1}|$, so the set family spans at least $(h-2)\log_2 a + 1$ levels. $\square$

## 2.3 Proof of Theorem 8

In this section we will give an upper bound on the size of the Lubell function of an induced $P$-free family. Lemma 2 holds for induced posets as well by an identical proof. Let $0 \leq a \leq b \leq n$. Let $\mathcal{H} \subset 2^{[n]}$ be a collection of sets which has the same number of sets, $N$, for each cardinality $i$ for $a \leq i \leq b$. Define $\alpha^{\#}(\mathcal{H}, P)$ to be the size of the largest subcollection of $\mathcal{H}$ containing no induced $P$.

**Lemma 8.** *Let $\mathcal{A}$ be an induced $P$-free family in $2^{[n]}$, in which the cardinality of every set is between $a$ and $b$. We have*

$$l_n(\mathcal{A}) \leq \frac{\alpha^{\#}(\mathcal{H}, P)}{N}.$$

*In particular, if $\mathcal{C}_k$ is an interval chain as defined in the Section 2.1, and $k \leq a$ and $b \leq n-k$ hold, we have*

$$l_n(\mathcal{A}) \leq \frac{\alpha^{\#}(\{A \in \mathcal{C}_k : a \leq |A| \leq b\}, P)}{2^{k-1}}.$$

*Proof.* The proof of Lemma 2 applies, observing that $a \le |A| \le b$. $\square$

We prove the following statement, which is slightly stronger than Theorem 8.

**Lemma 9.** *Let $P$ be a poset and let $c > \frac{1}{2}$. Let $n$ be a natural number, and let $0 \le a \le b \le n$. If $\mathcal{A}$ is an induced $P$-free family in which the cardinality of every set is between $a$ and $b$,*

$$l_n(\mathcal{A}) = \mathcal{O}\left((b-a)^c\right).$$

The following claim will be used recursively and is key to the proof of our lemma.

**Claim 1.** *If Lemma 9 holds for a given $c = c' > \frac{1}{2}$, then it also holds for $c = \frac{2c'}{2c'+1}$.*

*Proof of Claim.* Let $m = b-a+1$, and let $k = m^{\frac{2}{2c'+1}}$. Let $\mathcal{H} = \{A \in \mathcal{C}_k : a+k \le |A| \le b-k\}$. By definition $\mathcal{C}_k = \bigcup_{i=0}^{n-k} [A_i, A_{i+k}]$ (where $A_0 \subset A_1 \subset \ldots \subset A_n$ is an arbitrary maximal chain), and the levels $a+k$ to $b-k$ intersect $m-k$ of the intervals $[A_i, A_{i+k}]$. By substituting $k$ in the place of $n$ in Theorem 7, there is a constant $C$ such that $|\mathcal{A} \cap [A_i, A_{i+k}]| \le C\binom{k}{\lfloor \frac{k}{2} \rfloor}$ for every $i$. Thus $\alpha^{\#}(\mathcal{H}, P) \le (m-k)C\binom{k}{\lfloor \frac{k}{2} \rfloor} < Cm\binom{k}{\lfloor \frac{k}{2} \rfloor}$. By Lemma 8,

$$l_n(\{A \in \mathcal{C}_k : a+k \le |A| \le b-k\}) \le Cm\frac{\binom{k}{\lfloor \frac{k}{2} \rfloor}}{2^{k-1}} \le \frac{2\sqrt{2}}{\sqrt{\pi}}C\frac{m}{\sqrt{k}} = \frac{2\sqrt{2}}{\sqrt{\pi}}C\frac{m}{\sqrt{m^{\frac{2}{2c'+1}}}} = \frac{2\sqrt{2}}{\sqrt{\pi}}Cm^{\frac{2c'}{2c'+1}}. \tag{2.2}$$

By our assumption, using Lemma 9 with substituting $a+k-1$ in the place of $b$, we have

$$l_n(\{A \in \mathcal{C}_k : a \le |A| \le b-k-1\}) = \mathcal{O}\left(k^{c'}\right) = \mathcal{O}\left(m^{\frac{2c'}{2c'+1}}\right). \tag{2.3}$$

Similarly, by substituting $b-k+1$ in the place of $a$, we have

$$l_n(\{A \in \mathcal{C}_k : b-k+1 \le |A| \le b\}) = \mathcal{O}\left(m^{\frac{2c'}{2c'+1}}\right). \tag{2.4}$$

Adding up the inequalities (2.2), (2.3) and (2.4), we get

$$l_n(\{A \in \mathcal{C}_k : a \le |A| \le b\}) = \frac{2\sqrt{2}}{\sqrt{\pi}}Cm^{\frac{2c'}{2c'+1}} + 2\mathcal{O}\left(m^{\frac{2c'}{2c'+1}}\right) = \mathcal{O}\left((b-a)^{\frac{2c'}{2c'+1}}\right). \quad \square$$

*Proof of Lemma 9.* The lemma is trivial for $c = 1$. Substituting $c = 1$ in the proof of the claim directly gives a proof for $c = \frac{2}{3}$. Then, applying the claim recursively proves the statement for a sequence of exponents $c = c_i = \frac{2^i}{2^{i+1}-1}$. Indeed,

$$\frac{2c_i}{2c_i+1} = \frac{2\frac{2^i}{2^{i+1}-1}}{2\frac{2^i}{2^{i+1}-1}+1} = \frac{2^{i+1}}{2^{i+2}-1} = c_{i+1}.$$

The limit of the sequence is $\frac{1}{2}$, so it eventually becomes smaller than any $c > \frac{1}{2}$, proving our lemma. $\square$

# Chapter 3

# A method of decomposition of the cycle for forbidding subposets

Recall that the butterfly poset $B$, is defined by 4 elements $a, b, c, d$ with $a, b \leq c, d$. Our first new result is a strengthening of the following theorem about the Butterfly poset.

**Theorem 10** (De Bonis, Katona and Swanepoel [8]). *Let $n \geq 3$. Then, we have*

$$\mathrm{La}(n, B) = \Sigma(n, 2).$$

We introduce a poset $S$ which contains the butterfly as a strict subposet and prove that, nonetheless, the same bound holds. This poset, which we call the "skew"-butterfly, is defined by 5 elements, $a, b, c, d, e$, with $a, b \leq c, d$ and $b \leq e \leq d$ (see Figure 3).

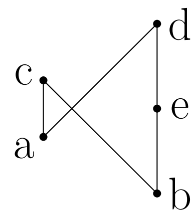


Figure 3.1: The skew-butterfly poset

**Theorem 11** (Methuku, Tompkins [24]). *Let $n \geq 3$, then we have*

$$\mathrm{La}(n, S) = \Sigma(n, 2).$$

A construction matching this bound is given by taking two consecutive middle levels of $2^{[n]}$. With this result (and all of the others) we also get the corresponding LYM-type inequality if we assume $\varnothing$ and $[n]$ are not in the family.

**Theorem 12** (Methuku, Tompkins [24]). *Let $n \geq 3$ and $\mathcal{A} \subset 2^{[n]}$ be a collection of sets not containing $S$ as a subposet, and assume that $\varnothing, [n] \notin \mathcal{A}$, then*

$$\sum_{A\in\mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 2.$$

For the proof of Theorem 12, we consider the set of intervals along a cyclic permutation (following Katona [15]). We partition these intervals into chains and consider the interactions of consecutive chains in the partition. The method and the proof of this result are given in Subsection 2.

We now mention some notable properties of $S$. It is one of the two posets whose Hasse diagram is a 5-cycle. The other is the harp, $H(4,3)$, and $\mathrm{La}(n, H(4,3))$ was determined exactly in the paper of Griggs, Li and Lu [11](the 4-cycles are $B$ and $D_2$). The skew-butterfly is contained in the $X$ $(a, b \leq c \leq d, e)$, a tree of height 3, like $B$, and so its asymptotics are determined by Bukh's theorem. The exact value of $\mathrm{La}(n, S)$ cannot be determined by the double chain method of Burcsi and Nagy [4] because one can find 5 sets on a double chain with no copy of $S$. Finally, if we subdivide any of the edges $ac, ad$ or $bc$ in the Hasse diagram of $S$, we get a poset for which there is a construction of size larger than $\Sigma(n,2)$.

Next, we consider a generalization of De Bonis, Katona and Swanepoel's theorem in a different direction. If instead of forbidding $B$, we forbid the pair of posets $Y$ and $Y'$ where $Y$ is the poset on 4 elements $w, x, y, z$ with $w \leq x \leq y, z$ and $Y'$ is the same poset but with all relations reversed, then $\mathrm{La}(n, Y, Y') = \mathrm{La}(n, B) = \Sigma(n, 2)$. This result is already implicit in the proof of De Bonis, Katona and Swanepoel. We extend the result by considering the posets $Y_k$ and $Y_k'$ defined by $k+2$ elements $x_1, x_2, \ldots, x_k, y, z$ with $x_1 \leq x_2 \leq \ldots \leq x_k \leq y, z$ and its reverse (so $Y = Y_2$ and $V = Y_1$). We prove

**Theorem 13** (Methuku, Tompkins [24]). *Let $k \geq 2$ and $n \geq k+1$, then*

$$\mathrm{La}(n, Y_k, Y_k') = \Sigma(n, k).$$

A construction matching this bound is given by taking $k$ consecutive middle levels of $2^{[n]}$. We also have the LYM-type inequality:

**Theorem 14** (Methuku, Tompkins [24]). *Let $k \geq 2$ and $n \geq k+1$. Assume that $\mathcal{A} \subset 2^{[n]}$ contains neither $Y_k$ nor $Y_k'$ as a subposet, and $\varnothing, [n] \notin \mathcal{A}$, then*

$$\sum_{A\in\mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq k.$$

We note that, again, the double chain method does not work for these pairs because one can have $2k+1$ sets on a double chain with no $Y_k$ and no $Y_k'$ by taking them consecutively on the secondary chain. We also note that, for this particular result, we can find another proof using the chain partitioning method of Griggs, Li and Lu [11] in addition to the approach described in this chapter.

Finally, we consider the more difficult induced case. We prove

**Theorem 15** (Methuku, Tompkins [24]). *For $n \geq 3$, we have*

$$\mathrm{La}^{\#}(n, Y, Y') = \Sigma(2, n).$$

We also have the LYM-type inequality:

**Theorem 16** (Methuku, Tompkins [24]). *Assume that $\mathcal{A} \subset 2^{[n]}$ contains neither $Y$ nor $Y'$ as an induced subposet, and $\varnothing, [n] \notin \mathcal{A}$, then*

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 2.$$

To prove Theorem 15 and Theorem 16, we introduce a second chain partitioning argument along the cycle. These partitions may be thought of as the analogue of orthogonal symmetric chain partitions for the cycle.

This chapter is organized as follows. In the first section we introduce the first chain decomposition and determine $\mathrm{La}(n, S)$. In the second section we use the same decomposition to find $\mathrm{La}(n, Y_k, Y_k')$ for all $k \geq 2$. In the last section we introduce the second decomposition and show that $\mathrm{La}^{\#}(n, Y, Y') = \Sigma(n, 2)$.

## 3.1 Forbidding $S$ and the first cycle decomposition

A cyclic permutation, $\sigma$, is a cyclic ordering $x_1, x_2, \ldots, x_n, x_1$ of the elements of $[n]$. We refer to the sets $\{x_i, x_{i+1}, \ldots, x_{i+t}\}$, with addition taken modulo $n$, as intervals along the cyclic permutation. For our purpose we will not consider $\varnothing$ or $[n]$ to be intervals. The following lemma is the essential ingredient of the proof of Theorem 12:

**Lemma 10.** *If $\mathcal{A}$ is a collection of intervals along a cyclic permutation $\sigma$ of $[n]$ which does not contain $S$ as a subposet, then*

$$|\mathcal{A}| \leq 2n.$$

To prove Lemma 10 we will work with a decomposition of the intervals along $\sigma$ into maximal chains. Set $\mathcal{C}_i = \{\{x_i\}, \{x_i, x_{i-1}\}, \{x_i, x_{i-1}, x_{i+1}\}, \ldots, \{x_i, x_{i-1}, \ldots, x_{i+n/2-1}\}\}$ when $n$ is even, and set $\mathcal{C}_i = \{\{x_i\}, \{x_i, x_{i-1}\}, \{x_i, x_{i-1}, x_{i+1}\}, \ldots, \{x_i, x_{i-1}, \ldots, x_{i-(n-1)/2}\}\}$ when $n$ is odd, where $1 \leq i \leq n$ (See Figure 3.2). Observe that the set of chains $\{\mathcal{C}_i\}_{i=1}^{n}$ forms a partition of the intervals along $\sigma$. We will refer to this partition as the *chain decomposition* of $\sigma$. Additionally, chains corresponding to consecutive elements of $\sigma$ are called consecutive chains.

If $\mathcal{A}$ does not contain $S$ as a subposet, and $\mathcal{C}$ is a chain from the chain decomposition of $\sigma$, then it is easy to see that $|\mathcal{A} \cap \mathcal{C}| \leq 4$. We will classify the chains in the chain decomposition by their intersection pattern with $\mathcal{A}$. If $|\mathcal{A} \cap \mathcal{C}| = k$, then we say $\mathcal{C}$ is of type $k$. When $k = 3$ we distinguish 3 cases (See Figure 3.3 for an example of each case). If $\mathcal{C}$ contains exactly 3 elements of $\mathcal{A}$, not all occurring consecutively on $\mathcal{C}$, then we say $\mathcal{C}$ is type $3^S$ ($S$ for separated). If $\mathcal{C}$ has exactly 3 elements of $\mathcal{A}$ occurring consecutively with two sets of odd

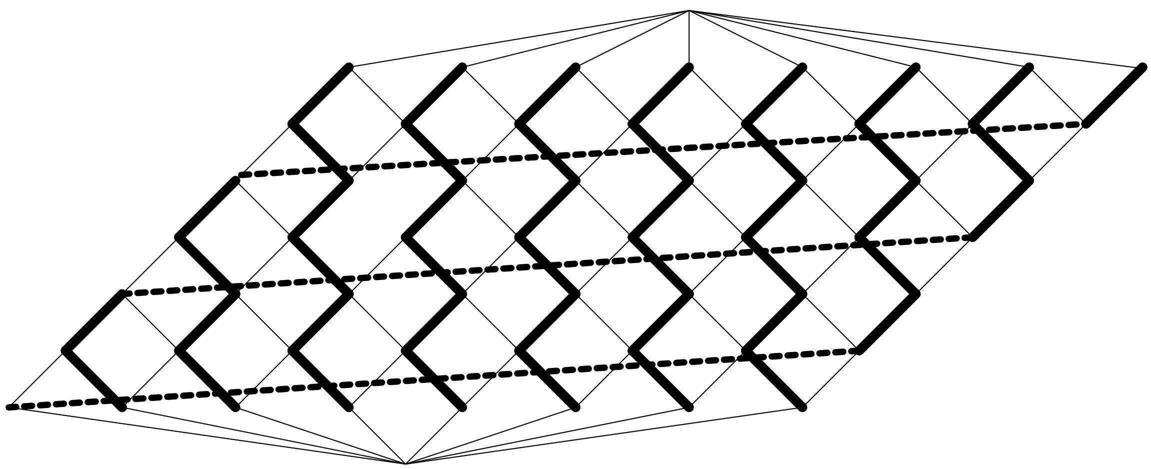

Figure 3.2: The chain decomposition is marked with bold lines on the poset of intervals along $\sigma$. The dashed lines indicate how the chains wrap around.

size, then $\mathcal{C}$ is type $3^R$ (facing right). If $\mathcal{C}$ has exactly 3 elements of $\mathcal{A}$ occurring consecutively with two sets of even size, then $\mathcal{C}$ is type $3^L$ (facing left).

We will now prove a sequence of lemmas showing which types of chains can occur consecutively in the chain decomposition of $\sigma$. These lemmas will let us disregard the exact intersection pattern of $\mathcal{A}$ with the chains and allow us to work instead with the sequence of chain types.

**Lemma 11.** *Let $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ be two consecutive chains in the chain decomposition of a cyclic permutation. If $\mathcal{C}_i$ is of type $4, 3^R$ or $3^S$, then $|\mathcal{A} \cap \mathcal{C}_{i+1}| \leq 1$.*

*Proof.* First, note that if $\mathcal{C}_i$ is of type 4, then we can remove a set from $\mathcal{A} \cap \mathcal{C}_i$ to make it type $3^S$. Hence, we may assume that $\mathcal{C}_i$ is of type $3^S$ or $3^R$.

In order to reduce case analysis, we will now argue that we only need to consider certain configurations of sets from $\mathcal{A}$ in $\mathcal{C}_i \cup \mathcal{C}_{i+1}$. Consider the Hasse diagram of $\mathcal{C}_i \cup \mathcal{C}_{i+1}$ as a graph (See Figure 3.4). Call the vertices corresponding to sets in $\mathcal{A}$ occupied and the rest unoccupied. If either the top or bottom vertex in the chain is occupied, then we extend $\mathcal{C}_i \cup \mathcal{C}_{i+1}$ in both directions maintaining the same relations between adjacent levels. Then, every occupied vertex either has degree 2 or degree 4. We will see that it is sufficient to consider the case when only degree 2 vertices are occupied. Indeed, if instead of taking a degree 4 vertex, we take an adjacent unoccupied degree 2 vertex, then no additional containments are introduced. If $\mathcal{C}_i$ is of type $3^R$ or $3^S$, then every occupied vertex of degree 4 can be replaced by a distinct adjacent unoccupied vertex of degree 2 (This cannot be done if $\mathcal{C}_i$ is type $3^L$). Thus, we may assume that all of the occupied vertices in $\mathcal{C}_i$ from the the Hasse diagram of $\mathcal{C}_i \cup \mathcal{C}_{i+1}$ have degree 2.

Let the sets in $\mathcal{A} \cap \mathcal{C}_i$ be $L, M$ and $N$ with $L \subset M \subset N$. Assume, by contradiction, that there are two sets $A, B \in \mathcal{A} \cap \mathcal{C}_{i+1}$ with $A \subset B$. We may assume that $A$ and $B$ correspond to degree 2 vertices in $\mathcal{C}_i \cap \mathcal{C}_{i+1}$. We will distinguish three cases by comparing the sizes of

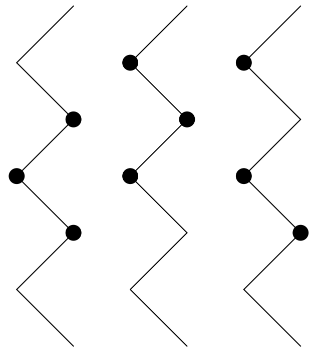


Figure 3.3: An example of chains of types $3^R$, $3^L$ and $3^S$ are drawn. The elements of $\mathcal{A} \cap \mathcal{C}$ are highlighted for each type.

$A$ and $B$ with the size of $M$. If $|A| < |M| < |B|$, then $L, M, N, A, B$ forms a skew-butterfly with $L, A \subset N, B$ and $L \subset M \subset N$. If $|M| < |A| < |B|$, then $L, M, N, A, B$ forms a skew-butterfly with $L, M \subset N, B$ and $L \subset A \subset B$. The case $|A| < |B| < |M|$ is symmetric. It follows that there can be at most one set in $\mathcal{A} \cap \mathcal{C}_{i+1}$.

□

**Lemma 12.** *Let $\mathcal{C}_i, \mathcal{C}_{i+1}$ and $\mathcal{C}_{i+2}$ be three consecutive chains in the chain decomposition of a cyclic permutation. If $\mathcal{C}_i$ is of type $4, 3^R$ or $3^S$ and $|\mathcal{A} \cap \mathcal{C}_{i+1}| = 1$, then $\mathcal{C}_{i+2}$ is of type $0, 1, 2$ or $3^R$.*

*Proof.* By contradiction, suppose $\mathcal{C}_i$ is type $4, 3^R$ or $3^S$, $|\mathcal{A} \cap \mathcal{C}_{i+1}| = 1$ and $\mathcal{C}_{i+2}$ is type $3^L, 3^S$ or 4. If $\mathcal{C}_i$ or $\mathcal{C}_{i+2}$ is of type 4, then we may disregard one set to make it type $3^S$. By similar reasoning as used in Lemma 11, we may assume all occupied vertices on the Hasse diagram of $\mathcal{C}_i \cup \mathcal{C}_{i+1} \cup \mathcal{C}_{i+2}$ from $\mathcal{C}_i$ and $\mathcal{C}_{i+2}$ have degree 2. Let $L, M, N$ be the three sets in $\mathcal{A} \cap \mathcal{C}_i$ in increasing order, and let $A, B, C$ be the three sets in $\mathcal{A} \cap \mathcal{C}_{i+2}$ in increasing order. Without loss of generality, we may assume $|M| > |B|$. This, in turn, implies that $|M| = |B| + 1$ for otherwise $L, M, N, A, B$ would be a skew-butterfly with $L, A \subset M, N$ and $A \subset B \subset N$. We will consider the possible locations of the set $S \in \mathcal{A} \cap \mathcal{C}_{i+1}$ on $\mathcal{C}_{i+1}$. If $|S| \leq |B|$, then $N, S, A, B, C$ is a skew-butterfly with $A, S \subset N, C$ and $A \subset B \subset C$. If $|S| > |B|$, then $L, M, N, S, A$ is a skew-butterfly with $L, A \subset N, S$ and $L \subset M \subset N$. Thus, in either case we have a contradiction. □

By symmetry, we also have the following corollaries of Lemmas 11 and 12:

**Corollary 3.** *Let $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ be two consecutive chains in the chain decomposition of a cyclic permutation. If $\mathcal{C}_{i+1}$ is of type $4, 3^L$ or $3^S$, then $|\mathcal{A} \cap \mathcal{C}_i| \leq 1$.*

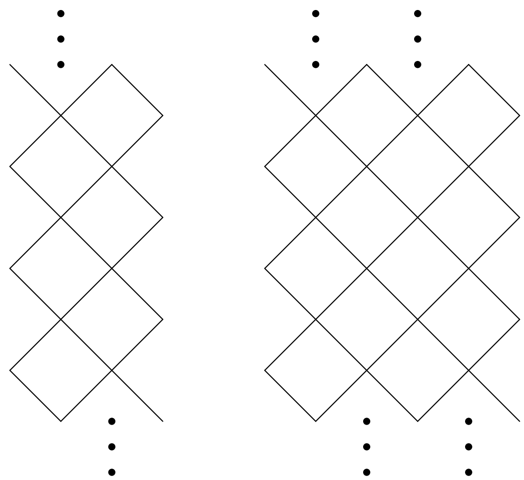

Figure 3.4: Hasse diagrams of $\mathcal{C}_i \cup \mathcal{C}_{i+1}$ and $\mathcal{C}_i \cup \mathcal{C}_{i+1} \cup \mathcal{C}_{i+2}$ are drawn.

**Corollary 4.** *Let $\mathcal{C}_i, \mathcal{C}_{i+1}$ and $\mathcal{C}_{i+2}$ be three consecutive chains in the chain decomposition of a cyclic permutation. If $\mathcal{C}_{i+2}$ is of type $4, 3^L$ or $3^S$ and $|\mathcal{A} \cap \mathcal{C}_{i+1}| = 1$, then $\mathcal{C}_i$ is of type $0, 1, 2$ or $3^L$.*

We now have sufficient information about which consecutive chain types are allowed to prove Lemma 10:

*Proof of Lemma 10.* We must show that the average intersection of $\mathcal{A}$ with chains from the decomposition is at most 2. To this end, we will form groups of chains such that the number of sets from $\mathcal{A}$ in each group is at most twice the size of that group.

First, consider chains of type 4. If there is a sequence of chains alternating between type 4 and type 0 spanning every chain in the chain decomposition, then it is easy to see that the average is at most 2. Otherwise, take each maximal group of consecutive chains alternating between type 0 and type 4, beginning and ending with a type 4 chain. Call such a group a 4-0-4 pattern (it may just consist of a single chain of type 4). If the group has length $\ell$, then there are $2\ell + 2$ sets contributed from $\mathcal{A}$. We will add additional chains to this group to decrease the average to 2. By Lemma 11, if the chain following the type 4 chain on either side is not type 0, then it must be type 1. In this case, we add the type 1 chain to the group. Otherwise, we have a type 0 chain followed by a chain of type 0,1,2 or 3. If it is type 3, we add both the type 0 and type 3 chain to our group. Otherwise, we just add the type 0 chain. In any case, if we have added $k$ more chains to our group (on both sides of the 4-0-4 pattern), then we have added a total of at most $2k - 2$ more sets from $\mathcal{A}$. Thus, in total, the group now consists of $k + \ell$ chains having at most $2k + 2\ell$ sets from $\mathcal{A}$, as desired.

Now, consider any remaining type 3 chain. Lemma 11 and Corollary 3 ensure that it has a type 1 or type 0 chain on at least one side (right or left). By Lemma 12 and Corollary 4 and by the previous grouping of the chains of type 4, we know that this chain was not used by any group consisting of chains of type 4. Thus, every type 3 chain may be grouped with its adjacent type 1 or 0 chain. All remaining chains in the decomposition have at most 2 sets from $\mathcal{A}$ and so we may group them all together.

□

We now derive the LYM-type inequality, Theorem 12, from Lemma 10.

*Proof.* We will double count pairs $(A, \sigma)$ where $A \in \mathcal{A}$ and $\sigma$ is a cyclic permutation of $[n]$. Let $f(A, \sigma)$ be the indicator function for $A \in \mathcal{A}$ and $A$ being an interval along $\sigma$. For each $A \in \mathcal{A}$, there are $|A|\,!\,(n-|A|)!$ cyclic permutations containing $A$ as an interval. It follows that

$$\sum_{A\in\mathcal{A}}\sum_{\sigma} f(A,\sigma) = \sum_{A\in\mathcal{A}} |A|\,!\,(n-|A|)!\,.$$

On the other hand, Lemma 10 implies

$$\sum_{\sigma}\sum_{A\in\mathcal{A}} f(A,\sigma) \le \sum_{\sigma} 2n = 2n!\,.$$

Dividing through by $n!$ gives

$$\sum_{A\in\mathcal{A}} \frac{1}{\binom{n}{|A|}} \le 2,$$

as desired. □

Finally, we deduce Theorem 11 from Theorem 12.

*Proof.* If $\mathcal{A}$ contains neither $[n]$ nor $\varnothing$, then the result follows easily from Theorem 11. If $\mathcal{A}$ contains $[n]$, but there is an $n-1$ element set $A$ not contained in $\mathcal{A}$, then replacing $[n]$ with $A$ in $\mathcal{A}$ introduces no new relations and so yields another family of the same size without a skew-butterfly. Thus, in this case, Theorem 11 again yields the result. If $\mathcal{A}$ contains $[n]$ and the entire $n-1^{st}$ level, let $\mathcal{A}' = \{A \in \mathcal{A} : |A| \le n-2\}$. Then, $\mathcal{A}'$ is an antichain, for otherwise we would have a skew-butterfly. Thus, $|\mathcal{A}'| \le \binom{n}{\lfloor n/2\rfloor}$ by Sperner's Theorem and so $|\mathcal{A}| \le \binom{n}{\lfloor n/2\rfloor}+n+1$. For $n \ge 5$ this implies $|\mathcal{A}| \le \binom{n}{\lfloor n/2\rfloor}+\binom{n}{\lfloor n/2\rfloor+1}$. An analogous argument works for the case when $\varnothing \in \mathcal{A}$. If $n = 4$ we give another argument (We are still assuming $\mathcal{A}$ contains all $n-1$ element sets). If $\mathcal{A}'$ is a full level, then $\mathcal{A}$ contains a skew-butterfly. If $\mathcal{A}'$ is not a full level, then the equality case of Sperner's theorem implies $|\mathcal{A}'| \le \binom{n}{\lfloor n/2\rfloor} - 1$, and so $|\mathcal{A}| \le n + \binom{n}{\lfloor n/2\rfloor}$ which yields the required bound when $n = 4$. The case $n = 3$ is easily checked by hand. □

We end this subsection by mentioning the relation between this approach and the double chain method. It is not hard to see that a double chain has the exact same poset structure as two consecutive chains in the chain decomposition described above. Namely, the degree 2 vertices from the Hasse diagram of consecutive chains correspond to the sets from the secondary chain of a double chain. It follows that any forbidden subposet result that can be determined exactly with the double chain method can also be determined exactly using a decomposition of a cyclic permutation, and, thus, chain decompositions of the cycle may be viewed as a generalization of the double chain method.

## 3.2 Forbidding $Y_k$ and $Y'_k$

We will use the same decomposition of the cycle as in Subsection 3.1. The new bound we must prove is

**Lemma 13.** *If $\mathcal{A}$ is a collection of intervals along a cyclic permutation $\sigma$ of $[n]$ which does not contain $Y_k$ or $Y'_k$ as a subposet, then*

$$|\mathcal{A}| \le kn.$$

As before, we will consider groups of consecutive chains. Each chain, $\mathcal{C}$, with $k+1$ sets in $\mathcal{C} \cap \mathcal{A}$ is characterized by whether the second largest element in $\mathcal{A} \cap \mathcal{C}$ faces left or faces right (has even or odd cardinality, respectively). We say that a chain with $k+1$ elements of $\mathcal{A}$ is of type $k+1^R$ if the second largest element faces right and $k+1^L$ if it faces left.

**Lemma 14.** *Let $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ be consecutive chains in the decomposition. If $\mathcal{C}_i$ is of type $k+1^R$, then $|\mathcal{A} \cap \mathcal{C}_{i+1}| \le k-1$, and $|\mathcal{A} \cap \mathcal{C}_{i+1}| = k-1$ implies that the largest element of $\mathcal{A} \cap \mathcal{C}_{i+1}$ is the same size as the second largest element of $\mathcal{A} \cap \mathcal{C}_i$.*

*Proof.* Let $A$ be the second smallest set in $\mathcal{A} \cap \mathcal{C}_i$ and $B$ be the second largest. Let $Y$ be the set of size $|B|$ in $\mathcal{C}_{i+1}$, and if $A$ is degree 2 (left), then let $X$ be the set of size $|A|-1$ in $\mathcal{C}_{i+1}$. If $A$ is degree 4, then let $X$ be the set of size $|A|$ in $\mathcal{C}_{i+1}$. In either case, let $\mathcal{R}$ be the collection of those sets in $\mathcal{C}_{i+1}$ (not necessarily in $\mathcal{A}$) having sizes strictly between $|X|$ and $|Y|$ (See Figure 3.5). Every set in $\mathcal{C}_{i+1} \cap \mathcal{A}$ must lie in $\mathcal{R} \cup \{X\} \cup \{Y\}$ for otherwise we would have a $Y_k$ or $Y'_k$. Now, $|\mathcal{A} \cap \mathcal{R}| \le k-2$ for otherwise we would have a $k+2$ chain (actually, $|\mathcal{A} \cap \mathcal{R}| \le k-3$ in the case $A$ is degree 4). If we take $k-1$ sets from $\mathcal{R} \cup \{X\}$, then we have a $Y'_k$ and so we can take at most $k-2$ sets total from $\mathcal{R} \cup \{X\}$. It follows that $|\mathcal{A} \cap \mathcal{C}_{i+1}| \le k-1$ with equality only if $Y \in \mathcal{A}$. □

By a symmetric argument we have

**Corollary 5.** *Let $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ be consecutive chains in the decomposition. If $\mathcal{C}_{i+1}$ is of type $k+1^L$, then $|\mathcal{A} \cap \mathcal{C}_i| \le k-1$, and $|\mathcal{A} \cap \mathcal{C}_i| = k-1$ implies that the largest element of $\mathcal{A} \cap \mathcal{C}_i$ is the same size as the second largest element of $\mathcal{A} \cap \mathcal{C}_{i+1}$.*

**Lemma 15.** *There are no 3 consecutive chains $\mathcal{C}_i, \mathcal{C}_{i+1}, \mathcal{C}_{i+2}$ such that $\mathcal{C}_i$ is type $k+1^R$, $\mathcal{C}_{i+1}$ is type $k-1$ and $\mathcal{C}_{i+1}$ is type $k+1^L$.*

*Proof.* Since $\mathcal{C}_i$ is type $k+1^R$ and $\mathcal{C}_{i+2}$ is type $k+1^L$, the respective second largest elements of $\mathcal{A} \cap \mathcal{C}_i$ and $\mathcal{A} \cap \mathcal{C}_{i+2}$ must be of different sizes. It follows from Lemma 14 and Corollary 5 that we can have at most $k-2$ sets in $\mathcal{A} \cap \mathcal{C}_{i+1}$. □

We now have what we need to prove Lemma 13.

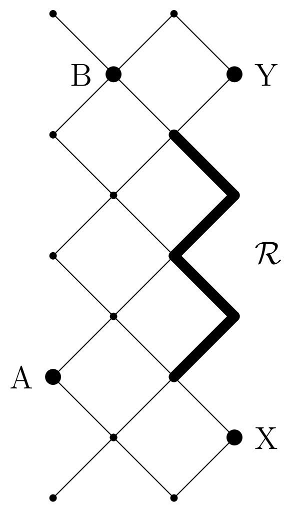


Figure 3.5: The sets $A$, $B$, $X$ and $Y$ are shown, and the collection $\mathcal{R}$ is marked in the case $A$ is degree 2.

*Proof of Lemma 13.* Every group of 3 consecutive chains of type $k+1^R$, $\leq k-2$ and $k+1^L$, respectively, may be grouped together yielding a total of at most $3k$ sets on 3 chains. All remaining chains of type $k+1^R$ may be paired with a chain of at most $k-1$ sets from $\mathcal{A}$ following it, and all remaining chains of type $k+1^L$ may be paired with a chain of at most $k-1$ sets preceding it. It follows that $\mathcal{A}$ consists of at most $kn$ intervals along the cyclic permutation $\sigma$. □

Theorem 14 follows directly from Lemma 13 as before. It remains to use Theorem 14 to deduce Theorem 13.

*Proof of Theorem 13.* Let $\mathcal{A} \subset 2^{[n]}$ be a $Y_k$ and $Y_k'$-free family. If neither of $\varnothing$ and $[n]$ are in $\mathcal{A}$, then the result is immediate from Theorem 14. If $\varnothing$ and $[n]$ are in $\mathcal{A}$, then $\mathcal{A} \setminus \{\varnothing, [n]\}$ is $k$-chain free and so has size at most $\Sigma(n, k-1)$ by Erdős's theorem. Since $2 + \Sigma(n, k-1) \leq \Sigma(n, k)$ for $n \geq k+1$ and $k \geq 2$, we are done. Finally, suppose that $\varnothing \in \mathcal{A}$ and $[n] \notin \mathcal{A}$. If there is a singleton set $\{x\} \notin \mathcal{A}$, then we may replace $\varnothing$ with $\{x\}$ and we are back in the first case. Hence, we may assume that $\mathcal{A}$ contains every singleton set ($\binom{[n]}{1} \subset \mathcal{A}$). Let $\mathcal{A}' = \mathcal{A} \setminus \{\{\varnothing\} \cup \binom{[n]}{1}\}$. Now, $\mathcal{A}'$ is $k$-chain free, so again by Erdős's theorem, $|\mathcal{A}'| \leq \Sigma(n, k-1)$. It follows that $|\mathcal{A}| \leq 1 + n + \Sigma(n, k-1)$. If $\mathcal{A}'$ contains $k-1$ full levels, then we have a copy of $Y_k'$, so we may assume we do not. However, then we may apply the equality case of Erdős's theorem to obtain that $|\mathcal{A}| \leq n + \Sigma(n, k-1)$. Finally, since $n \geq k+1$ implies that the $k^{th}$ largest level has size at least $n$, we have $|\mathcal{A}| \leq \Sigma(n, k)$, as desired. □

## 3.3 Forbidding induced $Y$ and $Y'$ and second cycle decomposition

As in the proof of Theorem 11, we will need to prove a lemma which bounds the largest intersection of an induced $Y$,$Y'$-free family with the set of intervals along a cyclic permutation.

**Lemma 16.** *If $\mathcal{A}$ is a collection of intervals along a cyclic permutation $\sigma$ of $[n]$ which does not contain $Y$ or $Y'$ as an induced subposet, then*

$$|\mathcal{A}| \leq 2n.$$

*Proof.* We will consider a different way of partitioning the chains along $\sigma$ from the one in the proofs of the previous theorems. Let $\sigma$ be the ordering $x_1, x_2, \ldots, x_n, x_1$. Group the intervals along $\sigma$ into chains $\mathcal{C}_i = \{\{x_i\}, \{x_i, x_{i+1}\}, \{x_i, x_{i+1}, x_{i+2}\}, \ldots, \{x_i, x_{i+1}, \ldots, x_{i+n-1}\}\}$ where $1 \leq i \leq n$. Observe that $\{\mathcal{C}_i\}_{i=1}^n$ is a partition of the intervals along $\sigma$.

We now consider a second way of partitioning the intervals by setting $\mathcal{C}'_i = \{\{x_i\}, \{x_i, x_{i-1}\}, \{x_i, x_{i-1}, x_{i-2}\}, \ldots, \{x_i, x_{i-1}, \ldots, x_{i-n+1}\}\}$ for $1 \leq i \leq n$. Observe that $\{\mathcal{C}'_i\}_{i=1}^n$ is again a partition (See Figure 3.6).

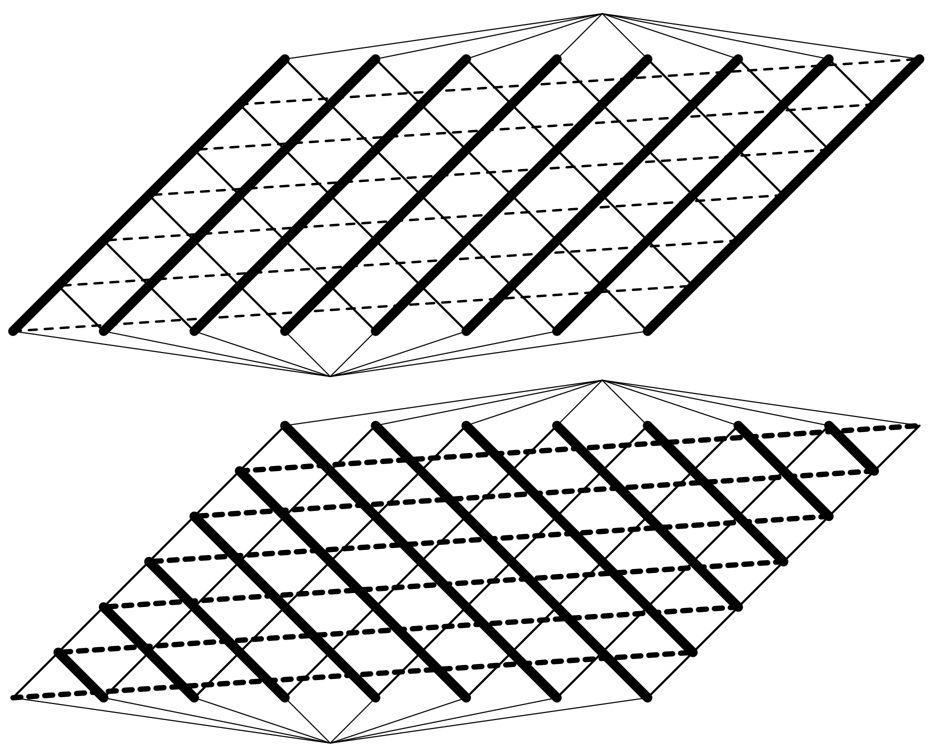

Figure 3.6: Orthogonal chain decompositions $\{\mathcal{C}_i\}_{i=1}^n$ (above) and $\{\mathcal{C}'_i\}_{i=1}^n$ (below) of the cycle are highlighted with bold lines. Dashed lines indicate how the chains wrap around.

Now, the two partitions we have defined have the property that if $A$ and $B$ are in $\mathcal{C}_i$ for some $i$, then at most one of $A$ and $B$ are in any $\mathcal{C}'_j$. Moreover, since each $A$ is contained in exactly one chain in each partition, it follows that each $A$ is contained in exactly 2 chains in the union of the two partitions. Thus, we have

$$\sum_{\mathcal{C}\in\{\mathcal{C}_i\}_{i=1}^n\cup\{C'_i\}_{i=1}^n} |\mathcal{A}\cap\mathcal{C}| = 2\,|\mathcal{A}|\,.$$

On the other hand, if a chain $\mathcal{C}\in\{\mathcal{C}_i\}_{i=1}^n$ intersects $\mathcal{A}$ in $k>2$ sets $A_1, A_2, \ldots, A_k$ with $A_1\subset A_2\subset\ldots\subset A_k$, then there are $k-2$ chains in $\mathcal{C}'\in\{\mathcal{C}'_i\}_{i=1}^n$ such that $|\mathcal{A}\cap\mathcal{C}'|=1$, namely those chains in $\{\mathcal{C}'_i\}_{i=1}^n$ containing $A_2, A_3, \ldots, A_{k-2}$ or $A_{k-1}$, as an intersection of greater than one would yield an induced $Y$ or $Y'$. Similarly, if a chain $\mathcal{C}'\in\{\mathcal{C}'_i\}_{i=1}^n$ intersects $\mathcal{A}$ in $k>2$ sets, then there are $k-2$ chains from $\{\mathcal{C}_i\}_{i=1}^n$ which intersect $\mathcal{A}$ in exactly one set. Here, we are using an additional property of the decomposition that if $A\in\mathcal{C}\cap\mathcal{C}'$, then no set larger than $A$ in $\mathcal{C}$ is comparable to a set larger than $A$ in $\mathcal{C}'$, and, similarly, no set smaller than $A$ in $\mathcal{C}$ is comparable to a set smaller than $A$ in $\mathcal{C}'$. We have shown that there is a total of $2k-2$ incidences of $\mathcal{A}$ with these $k-1$ chains. It follows that the number of pairs $(A,\mathcal{C})$ where $A\in\mathcal{A}, \mathcal{C}\in\{\mathcal{C}_i\}_{i=1}^n\cup\{\mathcal{C}'_i\}_{i=1}^n$ and $A\in\mathcal{C}$ is at most twice the number of chains. Thus,

$$\sum_{\mathcal{C}\in\{\mathcal{C}_i\}_{i=1}^n\cup\{C'_i\}_{i=1}^n} |\mathcal{A}\cap\mathcal{C}| \le 2\,|\{\mathcal{C}_i\}_{i=1}^n\cup\{\mathcal{C}'_i\}_{i=1}^n| = 4n.$$

Dividing through by 2 yields the desired inequality. □

Lemma 16 implies the LYM-type inequality, Theorem 16, exactly as in the previous proofs. It remains to derive the bound on $\mathrm{La}^{\#}(n, Y, Y')$ using Theorem 16.

*Proof of Theorem 15.* If $\mathcal{A}$ contains neither $\varnothing$ nor $[n]$, then we are done by Theorem 16. If $\varnothing$ and $[n]$ are in $\mathcal{A}$, then $\mathcal{A}\setminus\{\varnothing,[n]\}$ is induced $V$ and $\Lambda$ free. It follows from Katona and Tarjan [16] that

$$\mathrm{La}^{\#}(n, Y, Y') \le 2 + \mathrm{La}^{\#}(n, V, \Lambda) = 2 + 2\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor} \le \Sigma(n,2).$$

Now, assume without loss of generality that $\varnothing\notin\mathcal{A}$ but $[n]\in\mathcal{A}$, and let $\mathcal{A}'=\mathcal{A}\setminus\{[n]\}$. Since $\mathcal{A}'$ is induced $Y$ and $Y'$-free, it satisfies the hypothesis of Theorem 16. Assume, by contradiction, that $|\mathcal{A}'|=\Sigma(n,2)$. It follows that equality holds in Theorem 16. If $n$ is odd, then we have that $\mathcal{A}'=\binom{[n]}{\lfloor n/2\rfloor}\cup\binom{[n]}{\lceil n/2\rceil}$ which implies $\mathcal{A}$ induces a $Y'$, contradiction. If $n$ is even, then $\binom{[n]}{n/2}\subset\mathcal{A}'$ and $\binom{n}{n/2+1}$ sets from $\binom{[n]}{n/2-1}\cup\binom{[n]}{n/2+1}$ are in $\mathcal{A}$. Since $\mathcal{A}$ contains no $Y'$, it follows that $\mathcal{A}'\cap\binom{[n]}{n/2+1}=\varnothing$. Thus, we must have $\mathcal{A}'=\binom{[n]}{n/2-1}\cup\binom{[n]}{n/2}$, but then $\mathcal{A}$ still contains an induced $Y'$, contradiction. □

# Chapter 4

# Open problems and remarks

First, we mention an open problem that naturally arises from Theorem 6.

*Question* 1. Are there any posets $P$ for which equality holds in the following inequality when $k \geq 3$?

$$\mathrm{La}(n,P) \leq \frac{1}{2^{k-1}} \left(|P| + (3k-5)2^{k-2}(h(P)-1)-1\right) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

For $k = 2$, it is known [4] that there are infinitely many posets for which equality holds.

The most investigated poset for which even the asymptotic value of $La(n,P)$ has yet to be determined is the diamond $D_2$ (for example, [1, 11]). This poset is defined by four elements $\{a, b, c, d\}$ with the relations $a \leq b, c \leq d$. The best known upper bound is due to Kramer, Martin and Young [18] who proved a bound of $(2.25 + o(1))\binom{n}{\lfloor \frac{n}{2} \rfloor}$, the best possible bound using the Lubell function. It is conjectured that $\mathrm{La}(n, D_2)$ is asymptotic to $2\binom{n}{\lfloor \frac{n}{2} \rfloor}$. It was shown by Czabarka, Dutle, Johnston and Székely [6] that there are, in fact, many families of size larger than $\Sigma(n,2)$ so the asymptotic aspect of the conjecture is required. Better bounds were obtained in the case when the family is restricted to 3 levels including a bound of $2.208\binom{n}{\lfloor \frac{n}{2} \rfloor}$ by Axenovich, Manske and Martin [21], $2.1547\binom{n}{\lfloor \frac{n}{2} \rfloor}$by Manske and Shen [21] and $2.15121\binom{n}{\lfloor \frac{n}{2} \rfloor}$ by Balogh, Hu, Lidický and Liu [2]. In the induced version of the $D_2$-free problem an upper bound of $2.58\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is known [20].

Besides the diamond, there are still many posets for which the asymptotic value of $La(n,P)$ is not known. Some examples include the crown poset $O_{2t}$ defined by the relations $x_1 < y_1 > x_2 < y_2 \dots x_t < y_t > x_1$, the harp poset $\mathcal{H}(l_1, \dots, l_k)$ consisting of paths $P_{l_1}, \dots, P_{l_k}$ with their top elements identified and their bottom elements identified where $k \geq 1$ and $l_1 \geq \dots \geq l_k \geq 3$, generalized diamond poset $D_k := \mathcal{H}(3, \dots, 3)$ (i.e., each $l_i = 3$ for $1 \leq i \leq k$). For any poset $P$, Griggs and Lu [13] proposed the following conjecture:

**Conjecture 1** (Griggs, Lu [13]). *The limit* $\pi(P) := \lim_{n\to\infty} \frac{La(n,P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$ *exists and is an integer.*

We now list some well-known posets for which $\pi(P)$ hasn't been determined yet. Please note that we only list the best known bound to our knowledge and not the previous bounds and also, only the cases which haven't been settled yet. Since we do not know if $\pi(P)$ exists, to save space we just write $a \leq \pi(P) \leq b$ instead of

$$a \leq \liminf_{n\to\infty} \frac{La(n,P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq \limsup_{n\to\infty} \frac{La(n,P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq b.$$

| ***Poset*** $P$ | $\pi(P)$ |
|---|---|
| Crowns $O_6$ and $O_{10}$ | $1 \leq \pi \leq 1 + \frac{1}{\sqrt{2}}$ |
| Diamond $D_2$ | $2 \leq \pi \leq 2.25$ |
| Diamond (if the family is restricted to middle 3 levels) | $2 \leq \pi \leq 2.15121$ |
| Induced Diamond<br>(i.e., with relations $a < b < d$ and $a < c < d$ but $b$ and $c$ must be unrelated) | $2 \leq \pi \leq 2.58$ |
| Generalized Diamonds $D_k$ with $k \in [2^m - \binom{m}{\frac{m}{2}}, 2^m - 2]$<br>where $m := \lceil log_2(k+2) \rceil$ | $m \leq \pi < m + 1$ |
| Harp $\mathcal{H}(l_1, \ldots, l_k)$ where the path lengths $l_i$ are not all distinct | Open |

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