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Master's Thesis

Logarithmic Hodge Theory on Line Bundles

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Declaration of Authorship

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Abstract

This thesis intends to serve as an introduction to Hodge theory in the simplest possible setting: our base manifold is a compact Riemann surface Σ without boundary, the vector bundle $E \to \Sigma$ is the trivial complex line bundle. In this setup, the Betti, the de Rham and the Dolbeault groupoids are introduced and their equivalence is investigated.
The proof of the equivalence of the de Rham and Dolbeault groupoids uses the existence of harmonic metrics with respect to a connection D on E. The thesis concludes with the generalisation of the existence of such metrics to the case where the connection is no longer smooth but has logarithmic singularities, and the weight of the associated local system vanishes.

Contents

Introduction

Hodge theory is a fastly developing branch of modern geometry, and it is deeply related to (at least) two of the Clay Institute's Millennium Problems. In recent years irregular Hodge theory has been widely investigated by the mathematics and mathematical physics communities. The general, non-abelian theory has been explored by many, see [Witt], [GMN], [Kon], [Boa], [Moc], [Sab], etc. Although these expositions use some heavy machinery, the regular (without singularities) abelian theory on a compact Riemann surface is classic, and is very well-exposed using today's terminology in [GoXi].

Yet, as far as we know, the abelian theory with singularities on a non-compact Riemann surface has not been treated in the literature. The aim of the present thesis is to generalise the results covered in [GoXi] to the logarithmic case.

The moduli spaces underlying non-abelian Hodge theory have rich geometric structures. In the abelian case, one may explicitly check how these structures behave under the non-abelian Hodge correspondence. In the non-abelian case however, much of the hands-on understanding of these spaces is lacunary, and the transformation behaviour of the geometric structures is, at some points, merely conjectural. This thesis extends some of the geometric understanding of these spaces from the smooth abelian case to the logarithmic case, with the hope of getting some ideas about the non-abelian case too.

Chapter 1 is a summary of the theoretical background and description of tools that we use in the rest of the thesis, including connections on principal bundles and vector bundles, complex differential geometry, the Hodge decomposition theorem and sheaf theory.

Chapter 2 serves as an overview of the regular abelian theory. This means a short description of the category theoretic equivalence between the

(B) Betti groupoid of the representations of the fundamental group, $(\mathrm{Hom}(\pi_1(\Sigma), \mathbb{C}^*), \mathbb{C}^*)$

(dR) de Rham groupoid of flat connections modulo linear gauge transformations, $(\mathcal{F}_l(E), \mathcal{G}_l(E))$

(D) Dolbeault groupoid of Higgs pairs modulo linear gauge transformations, $(\mathrm{Higgs}(E), \mathcal{G}_l(E))$

The proof of the equivalence of (dR) and (D) relies on the existence of harmonic metrics on the bundle.

Chapter 3 generalises the existence of such metrics to the case where the connection has logarithmic singularities which is a more direct proof than using the general non-abelian theory.

Chapter 1

Preliminaries

1.1 Fiber bundles

Definition 1.1. Let B, F be smooth manifolds. A (smooth) **fiber bundle** over B with fiber F is a manifold E and a smooth map $\pi : E \to B$, if for every $b \in B$ there exists a neighbourhood $U_b \subset B$ and a diffeomorphism $\phi_b : \pi^{-1}(U_b) \to U_b \times F$ such that the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U_b) \arrow[r, "\phi_b"] \arrow[dr, "\pi"'] & U_b \times F \arrow[d, "proj_1"] \\
& U_b
\end{tikzcd}$$

Example 1.2 (Trivial Bundle)**.** Let $E = B \times F$ the product manifold of B and F, and $\pi = proj_B$. This is clearly a fiber bundle, for each $b \in B$ we can set $U_b = B$ and $\phi_b = id_{B\times M}$.

It follows from the definition that for each $b \in B$, the preimage $E_b := \pi^{-1}(b)$ is diffeomorphic to F. We call E_b the **fiber** over b. By restricting the bundle to U_b we obtain a trivial bundle, therefore the neighbourhood U_b is called a trivialising neighbourhood of b. For two overlapping trivialisations $(U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta)$ and $b \in U_\alpha \cap U_\beta$ the composition $f \to proj_2 \circ \phi_\alpha \circ \phi_\beta^{-1}(b, f)$ is a diffeomorphism of the fiber $\pi^{-1}(b)$.

Example 1.3 (Möbius strip)**.** The Möbius strip is nontrivial fiber bundle over S^1 with fiber $[0, 1]$ that can be obtained as

$$[0, 1] \times [0, 1]/\sim$$

where $\sim$ is given by $(0, y) \sim (1, 1 - y)$.

Definition 1.4. If F is an n-dimensional linear space over $\mathbb{K}$ and for any two overlapping trivialisations (U_α, ϕ_α) and (U_β, ϕ_β) the map $v \to proj_2 \circ \phi_\alpha \circ \phi_\beta^{-1}(b, v)$ is a $\mathbb{K}$-linear isomorphism for every $b \in U_\alpha \cap U_\beta$, then the fiber bundle is called a **vector bundle** of rank n.

One of the most important examples of a vector bundle is the tangent bundle of a smooth manifold.

Example 1.5. The **tangent bundle** of an n-dimensional smooth manifold M is the disjoint union

$$TM = \bigsqcup_{p\in M} T_pM = \{(p, v) \mid p \in M, v \in T_pM\}$$

of its tangent spaces. To put a topology on TM, let (U_i, α_i) be the atlas of M, $\{\partial_{p,1}, ..., \partial_{p,n}\}$ a basis for T_pM that depends smoothly on $p \in U_i$, and define $\pi : TM \to M$ as $\pi(p, v) = p$. The maps $\bar{\alpha}_i(p, \sum v^j \partial_{p,j})) = (\alpha_i(p), v^1, ..., v^n)$ are diffeomorphisms between $\pi^{-1}(U_i)$ and $\mathbb{R}^{2n}$ and can be used to put the topology on TM. This makes TM a 2n-dimensional smooth manifold.

Let $\pi : E \to B$ be a vector bundle of rank n and $(U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta)$ two overlapping local trivialisations. Then the map $\phi_\alpha \circ \phi_\beta^{-1}$ is defined on $U_\alpha \cap U_\beta \times \mathbb{R}^n$ and satisfies $\phi_\alpha \circ \phi_\beta^{-1}(x, v) = (x, \phi_{\alpha\beta}(v))$ for some $\phi_{\alpha\beta} \in GL(n, \mathbb{R})$. The function $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$ smoothly depends on the base point, and is called the transition function from (U_β, ϕ_β) to (U_α, ϕ_α).
Given the local trivialisation (U_i, ϕ_i) the transition functions give us the way to patch these local trivialisations together and obtain the total space of the vector bundle. These transition functions clearly satisfy $\phi_{\alpha\beta} = \phi_{\beta\alpha}^{-1}$ and the cocycle condition

$$\phi_{\alpha\beta}\phi_{\beta\gamma}\phi_{\gamma\alpha} = id_{U_\alpha \cap U_\beta \cap U_\gamma}$$

Using the above observations, we can build a vector bundle over B if the local trivialisations and the transition functions are given.

Example 1.6. For the tangent bundle the transition functions are simply the Jacobians

$$\phi_{\alpha\beta} = D(\phi_\alpha \circ \phi_\beta^{-1})$$

Given two vector spaces A and B, we can form the following new vector spaces:

$$A \oplus B \qquad A \otimes B \qquad \mathrm{Hom}(A, B) \qquad A^* \qquad \bigwedge^k A$$

We want to do the the same for vector bundles fiberwise. This can be done by the the following technique.
Let $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ be vector bundles over the same space B. Let $E_b = F(E_1, E_2)$ be one of

$$(E_1 \oplus E_2)_b \qquad (E_1 \otimes E_2)_b \qquad (\mathrm{Hom}(E_1, E_2))_b \qquad (E_1^*)_b \qquad (\bigwedge^k E_1)_b$$

for every $b \in B$.
In the following paragraph we slightly modify the notation, and let $(U_i, id_{U_i} \times \phi_i)$ denote the local trivialisations, where $\phi_i : U_i \to \mathbb{R}^n$. (We originally defined ϕ_i to be a map from $\pi^{-1}(U_i)$ to $U_i \times \mathbb{R}^n$.) So we replace ϕ_i by its graph $x \to (x, \phi_i(x))$.
If $(U_{1,\alpha}, id_{U_{1,\alpha}} \times \phi_{1,\alpha})$ are the local trivialisation of $\pi_1 : E_1 \to B$ glued together by $\phi_{1,\alpha\beta}$ and $(U_{2,\gamma}, id_{U_{\gamma,j}} \times \phi_{2,\gamma})$ are the local trivialisation of $\pi_2 : E_2 \to B$ glued together by $\phi_{2,\gamma\delta}$, then $(U_{1,\alpha} \cap U_{2,\gamma}, id_{U_{1,\alpha} \cap U_{2,\gamma}} \times F(\phi_{1,\alpha}, \phi_{2,\gamma}))$ are the local trivialisations of $F(E_1, E_2)$ glued together by $F(\phi_{1,\alpha\beta}, \phi_{2,\gamma\delta})$.

Definition 1.7. Let E and F be vector bundles over the same space B. A **homomorphism** of vector bundles $f : E \to F$ is a smooth map between the total spaces that preserves the fibers and is linear on them. If f is a diffeomorphism and a linear isomorphism on the fibers, then f is said to be an **isomorphism** of vector bundles.

Definition 1.8. A **section** of a bundle $\pi : E \to B$ is a smooth map $s : B \to E$ such that $\pi \circ s = id_B$. We denote the space of all sections of E by $\Gamma(E)$.

Example 1.9. The sections of the tangent bundle are the smooth vector fields on the base manifold.

Example 1.10. Every vector bundle admits a section, the so-called zero section, $s(b) = 0$ for every $b \in B$. For a vector bundle of rank n, n nowhere dependent (local) sections determine a (local) trivialisation of the bundle.

Definition 1.11. Let G be a Lie group. A **principal** G-**bundle** over B is a fiber bundle $\pi : E \to B$ and a smooth right action R of G on E satisfying the following:

- Each R_g preserves the fibers and acts freely and transitively on them
- There is an open cover $\{U_\alpha\}$ of B with local trivialisations ($\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$) such that the group action commutes with the trivialisations

$$\phi_\alpha(p) = (\pi(p), a) \Rightarrow \phi_\alpha(R_g(p)) = (\pi(p), ag)$$

It follows from the first condition that the fibers are diffeomorphic to G. The second condition implies that the transition maps between local trivialisations also commute with the group action and therefore

$$\phi_\alpha \circ \phi_\beta^{-1}(b, g) = (\phi_\alpha \circ \phi_\beta^{-1}(b, e))g = (b, \phi_{\alpha\beta}(e))g = (b, L_{\phi_{\alpha\beta}(e)}(g))$$

i.e. the transition maps are left actions of G.

1.2 Connections on principal bundles

In what follows let $\pi : P \to B$ be a principal G-bundle.

Connections as horizontal subspace distributions

Definition 1.12. Let $p \in P$. The **vertical space** at p is defined as

$$V_pP = ker(d\pi)$$

Intuitively, V_pP is the vector subspace of T_pP that is parallel to the fiber. The complementary subspaces of V_pP are called **horizontal**. While the vertical subspace is uniquely defined by the projection, we have many choices for the horizontal complement, connections are introduced to deal with this matter.

Definition 1.13. A **connection** on $\pi : P \to B$ is a smooth assignment of a horizontal subspace H_pP to each $p \in P$ such that R_g preserves the horizontal subspaces, i.e.

$$H_{R_g(p)}P = (dR_g)_p H_pP$$

for all $g \in G$.

Connections can be defined on any fiber bundle $\pi : E \to B$ as smooth horizontal subspace distributions (for principal bundles we also require this distribution to be compatible with the group action). Given a connection, a tangent vector $X \in T_bB$ to the base manifold B, and a point $p \in E_b$ of the fiber over b, we can lift X to a tangent vector $\tilde{X} \in H_pE \subset T_pE$. This is doable because dim $H_pE =$ dim T_pB and the projection induces an isomorphism between these spaces. The reason for doing all this is that two fibers over different points, say E_a and E_b, are diffeomorphic by definition, but not in a natural way. If B is path connected, then by picking a curve $\gamma \in B$ from a to b, lifting $\gamma' \in TB$ to $\tilde{\gamma'} \in HE \subset TE$ with initial point $\tilde{a} \in E_a$ and solving this differential equation yields a diffeomorphism from E_a to E_b. Hence the name connection. This diffeomorphism, of course, depends on the connection itself and the chosen curve γ. This construction transports the points of E_a to every fiber over γ, and is called the **parallel transport** along γ.

Connections as Lie-algebra-valued 1-forms

An alternative description of connections can be given, but we need a few more definitions for that.

Definition 1.14. The **Lie-algebra** $\mathfrak{g}$ of a Lie-group G is the set of left-invariant vector fields on G, i.e.

$$X \in \mathfrak{g} \Leftrightarrow X = (dL_g)X$$

for all $g \in G$.

The Lie-algebra $\mathfrak{g}$ can be identified with T_eG by $X \leftrightarrow X(e)$. Every left-invariant vector field $X \in \mathfrak{g}$ is complete (because every integral curve can be extended by the left-invariance of X), therefore the following definition makes sense:

Definition 1.15. The exponential map $\exp : \mathfrak{g} \to G$ is given by

$$\exp(X) = \gamma(1)$$

where $\gamma(t)$ is the integral curve of X for which $\gamma(0) = e$.

It follows by the chain rule that $\exp(tX) = \gamma(t)$.

Definition 1.16. The **fundamental vector field** corresponding to $X \in \mathfrak{g}$ is defined as

$$X_p^* = \frac{d}{dt}\Big|_{t=0} R_{\exp(tX)}(p)$$

The right action of G preserves the fibers, therefore $X_p^* \in V_pP$. Suppose $X_p^* = 0$ for some $X \in \mathfrak{g}$. Then p is a fixed point of $R_{\exp(tX)}$ for all t. But G acts freely on E, so $\exp(tX) = e \Rightarrow X = 0$. This shows that the map $X \to X_p^*$ is a linear isomorphism from $\mathfrak{g}$ to V_pP.
Moreover, this linear isomorphism turns out to be a Lie algebra isomorphism. To see this, consider the map $\alpha_p : G \to P$ given by $\alpha_p(g) = pg$, i.e.the right action of $g \in G$ on $p \in P$. The group law $p(g_1g_2) = (pg_1)g_2$ becomes

$$\alpha_p \circ L_{g_1}(g_2) = \alpha_{pg_1}(g_2)$$

Taking differentials at $g_2 = e$,

$$(d\alpha_p)_{g_1} \circ (dL_{g_1})_e(X_e) = (d\alpha_{pg_1})_e(X_e)$$

for any $X \in \mathfrak{g}$. Using this,

$$X_{pg_1}^* = (d\alpha_{pg_1})_e(X_e) = (d\alpha_p)_{g_1} \circ (dL_{g_1})_e(X_e) = (d\alpha_p)_{g_1}(X_{g_1})$$

where the last equality comes from the left invariance of X. Thus the vector field $X^*_{|P_{\pi(p)}}$ is the pushforward of the vector field X along α_p. Since $d\alpha_p$ commutes with the Lie bracket, so does the map $X \to X^*$. This paragraph justifies the

$$[A^*, B^*] = [A, B]^*$$

type equalities in the upcoming proofs.

Now we can give the alternative definition of a connection. Fix $p \in P$. Every choice of H_pP defines a projection $T_pP \to V_pP \cong \mathfrak{g}$, which is in fact a $\mathfrak{g}$-valued 1-form on P, called the connection 1-form of the connection.

Theorem 1.17. *If ω is a connection 1-form, then it satisfies the following:*

- *ω vanishes on horizontal vectors*
- *$\omega(X^*) = X$ for all $X \in \mathfrak{g}$*
- *$\omega(dR_gX) = Ad_{g^{-1}}\omega(X)$*

Proof. The first two claims follow from the definition of ω. For the third we decompose X as the sum of a vertical and a horizontal vector, $X = V + H$. By the linearity of ω it is enough to prove the statement for V and H separately. For the horizontal component we have

$$\omega(dR_gH) = 0 = Ad_{g^{-1}}\omega(H)$$

because both H and dR_gH are horizontal. For the vertical component there exists $A \in \mathfrak{g}$ such that $V = A^*$, then

$$\begin{aligned}
\omega_{R_g(p)}(dR_gV_p) = \omega_{R_g(p)}(dR_gA_p^*) &= \omega_{R_g(p)}\Big(\frac{d}{dt}\Big|_{t=0} R_g \circ R_{exp(tA)}(p)\Big) \\
&= \omega_{R_g(p)}\Big(\frac{d}{dt}\Big|_{t=0} R_g \circ R_{exp(tA)} \circ R_{g^{-1}}(R_g(p))\Big) \\
&= \omega_{R_g(p)}\Big(\frac{d}{dt}\Big|_{t=0} R_{g^{-1}exp(tA)g}(R_g(p))\Big) \\
&= \omega_{R_g(p)}((Ad_{g^{-1}}A)^*_{R_g(p)}) \\
&= (Ad_{g^{-1}}A)_{R_g(p)} \\
&= Ad_{g^{-1}}\omega(V)_{R_g(p)}
\end{aligned}$$

□

Conversely, if a 1-form ω on P satisfies the conditions of Theorem 1.17, then it defines a connection on P by $H_pP = ker(\omega_p)$.

Remark 1.18. By the compatibility of the connection and the group action, the connection at one point determines the connection at every point along the fiber. Namely, given ω_p, we can use the group action to spread it out along the fiber to obtain $\omega_{R_g(p)}$.
This also means that for a collection of local trivialisations $\{U_i, \phi_i\}$, we can pull back ω along ϕ_i to obtain a $\mathfrak{g}$-valued 1-form on B.

1.3 Integrability of connections

Definition 1.19. Let $h : T_pP \to H_pP$ be the projection to the horizontal subspace, and α an n-form. Define the $(n+1)$-form

$$D\alpha(X_1, ..., X_n) = d\alpha(hX_1, ..., hX_n)$$

In particular the **curvature 2-form** of a connection 1-form ω is defined as

$$\Omega(X, Y) = D\omega(X, Y) = d\omega(hX, hY)$$

If $\Omega = 0$, then ω is said to be **flat**.

Theorem 1.20 (Structure equation)**.** *Let ω be a connection 1-form and Ω is its curvature 2-form. Then*

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]$$

where d is the exterior derivative, and $[\omega \wedge \omega]$ is defined as

$$[\omega \wedge \omega](X, Y) = [\omega(X), \omega(Y)] - [\omega(Y), \omega(X)] = 2[\omega(X), \omega(Y)]$$

Proof. We need to show that $d\omega(hX, hY) = d\omega(X, Y) + [\omega(X), \omega(Y)]$.
First, suppose X and Y are horizontal. Then $hX = X, hY = Y$ and $\omega(X) = \omega(Y) = 0$. The structure equation reduces to

$$d\omega(X, Y) = d\omega(X, Y)$$

Now let $X = A^*$ and $Y = B^*$ be vertical and $A, B \in \mathfrak{g}$. Then $hX = hY = 0$, which implies $d\omega(hX, hY) = 0$. For the RHS we have

$$\begin{aligned} d\omega(X, Y) + [\omega(X), \omega(Y)] &= d\omega(A^*, B^*) + [A, B] \\ &= A^*\omega(B^*) - B^*\omega(A^*) - \omega\big([A^*, B^*]\big) + [A, B] \\ &= -\omega\big([A^*, B^*]\big) + [A, B] \\ &= -[\omega(A^*), \omega(B^*)] + [A, B] = 0 \end{aligned}$$

Finally, let $X = A^*$ be vertical and Y horizontal. Then $d\omega(hX, hY) = d\omega(0, hY) = 0$ and $[\omega(X), \omega(Y)] = [A, 0] = 0$. We only need to show that $d\omega(X, Y) = 0$.

$$\begin{aligned} d\omega(X, Y) = d\omega(A^*, Y) &= A^*\omega(Y) - Y(\omega(A^*)) - \omega([A^*, Y]) \\ &= -\omega([A^*, Y]) \\ &= -\omega\Big(\lim_{t\to 0}\frac{1}{t}(Y_{\Phi^t_{A^*}} - Y)\Big) \end{aligned}$$

where $Y_{\Phi^t_{A^*}}$ is the pullback of $Y_{A^*(t)}$ along the flow of A^*. The vector field Y is a horizontal, $Y_{\Phi^t_{A^*}}$ is also horizontal, because the right action of the group preserves horizontal subspaces. The above expression is then 0 by the vanishing of ω on horizontal vectors.

The general case follow from the above 3 special cases by linearity. □

Theorem 1.21 (Bianchi's identity)**.** *Let Ω be a curvature 2-form of a connection. Then $D\Omega = 0$.*

Proof. By the definition of D it is enough to prove the statement for horizontal vectors X, Y, Z. Then by the structure equation

$$\begin{aligned} D\Omega(X, Y, Z) &= d^2\omega(X, Y, Z) + \frac{1}{2}d[\omega \wedge \omega](X, Y, Z) \\ &= 0 + d[\omega(X), \omega(Y)](Z) \\ &= d\omega(X, Z)\omega(Y) - \omega(X)d\omega(Y, Z) \end{aligned}$$

which is 0, because ω vanishes on horizontal vectors. □

Definition 1.22. Let M be a smooth manifold. A k-dimensional **subspace distribution** δ is an assignment of a k-dimensional subspace of T_pM to each point $p \in M$.

Example 1.23. A connection is a horizontal distribution on the total space of the principal bundle.

Definition 1.24. A k-dimensional distribution δ on M is said to be **integrable** if for every $p \in M$ there exists a k-dimensional submanifold N of M such that $p \in N$ and N is everywhere tangent to δ.

Theorem 1.25 (Frobenius)**.** *A k-dimensional distribution δ on M is integrable if and only if $[\delta, \delta] \subset \delta$.*

Where $[\delta, \delta] \subset \delta$ means that if X, Y are vector fields on M that are contained in D, then so is $[X, Y]$.

Proof. [LeeSM], Theorem 19.12 □

For connections this condition becomes equivalent to vanishing of its curvature 2-form.

Theorem 1.26. *A connection is integrable if and only if it is flat.*

Proof. This is just a simple computation:

$$\begin{aligned}\Omega(X, Y) &= d\omega(hX, hY) \\ &= hX\big(\omega(hY)\big) - hY\big(\omega(hX)\big) - \omega\big([hX, hY]\big) \\ &= -\omega\big([hX, hY]\big)\end{aligned}$$

And the theorem follows by the theorem of Frobenius. □

1.4 Digression to vector bundles

Now that we have developed some machinery on principal bundles, we derive similar results for vector bundles.

Connections on vector bundles

Definition 1.27. A **linear connection** on a vector bundle $E \to M$ is a smooth horizontal subspace distribution that satisfies

$$G_*(H_v E) = H_{Gv} E$$

where G is a linear automorphism of the bundle.

Equivalently, we can define connections as covariant derivations.

Definition 1.28. A **covariant derivative** on a vector bundle $E \to M$ is a linear map $\nabla : \Gamma(TM) \to \Gamma(T^*M \otimes E)$ satisfying a Leibniz rule

$$\nabla_X(fs) = df(X)s + f\nabla_X(s)$$

for $f \in C^\infty(M), X \in \Gamma(TM), s \in \Gamma(E)$.

Given a covariant derivative ∇, the corresponding horizontal subspace distribution is given by

$$H_{s(b)}E = \{s_*(X) - \nabla_X(s) | X \in T_b M\}$$

this distribution is well-defined, i.e. $H_{s(b)} = H_{s'(b)}$ if $s(b) = s'(b)$. Conversely, given a horizontal subspace distribution, the corresponding covariant derivative is

$$\nabla_X(s) = s_*(X) - \mathrm{Hor}_{s(b)}(X)$$

where $\mathrm{Hor}_{s(b)}(X)$ is the horizontal lift of $X \in T_bM$ to $H_{s(b)}E$. In the previous formulas the fiber E_b and the vertical subspace $V_{s(b)}E$ are identified by the isomorphism

$$E_b \to V_{s(b)}E : w \to \gamma(t) = s(b) + tw$$

Now let $\{E_1, E_2, ..., E_n\}$ be a local trivialisation. Every section s can be written as $s = s^i E_i$ (using the summation convention) for some $s^i \in C^\infty(M)$. Using the definition of the covariant derivative, we have

$$\nabla_X(s) = \nabla_X(s^i E_i) = ds^i(X)E_i + s^i \nabla_X(E_i) = ds(X) + As$$

where A is a matrix of 1-forms whose ith column is formed by the components $\nabla_X E_i$ in the basis $\{E_1, ..., E_n\}$. Therefore we can write

$$\nabla = d + A$$

Conversely, given any matrix of 1-forms A defines a connection $d + A$.
Now let $\{E'_1, ..., E'_n\}$ be an other basis, then we can write $E_i = T_i^j E'_j$ for some matrix T. The covariant derivative ∇ then acts as

$$\nabla_X(s) = \nabla(s^i E_i) = \nabla_X(s^i T_i^j E'_j) = d(s^i T_i^j)(X)E'_j + s^i T_i^j \nabla_X(E'_j)$$

and we have $\nabla = d + A'$ where A' is a matrix of 1-forms in the basis $\{E'_1, ..., E'_n\}$.

Definition 1.29. We define the **curvature** $F(\nabla)$ of a linear connection $\nabla = d + A$ as the 2-form

$$dA(X, Y) + A \wedge A(X, Y) = dA(X, Y) + A(X)A(Y) - A(Y)A(X)$$

The associated bundle

Now let $P \to M$ be a principal G-bundle, V a vector space and $\rho : G \to GL(V)$ a representation of G.

Definition 1.30. Let $P \to M$ be defined by trivialisations $\{U_i, \phi_i\}$ and transition functions $\phi_{\alpha\beta}$ corresponding to the left action of $g_{\alpha\beta}$. The **associated bundle** $E \to M$ is the vector bundle defined by the set of trivial bundles $U_i \times V$ and transition maps $\rho(g_{\alpha\beta})$.

The derivative of ρ at the identity is a map from the Lie algebra of G to the endomorphism group of V,

$$\rho_* : \mathfrak{g} \to End(V)$$

A connection 1-form ω on $P \to M$ can be thought of a as a $\mathfrak{g}$-valued 1-form on M (see Remark 1.18), its composition with ρ_* is an endomorphism valued 1-form $\rho_* \circ \omega$ on M. This endomorphism valued 1-form is the linear connection induced by ω.

Frame bundles

Let $E \to M$ be a vector bundle. We can associate a principal bundle $F(E) \to M$, the so-called **frame bundle** of $E \to M$ as follows. Let the fiber $F(E)_b$ be the set of all ordered bases of E_x with the action of $GL(E_x)$ as basis transformations. This space can be given a natural topology and bundle structure using those of $E \to M$. The structure group is a linear group by definition, and the vector bundle associated to $F(E) \to M$ is $E \to M$.

Moreover, since the representation is the identity, its derivative is bijective, therefore every connection on the vector bundle is induced by a connection on its frame bundle. Also, we have

$$\rho_* \circ (d\omega + \omega \wedge \omega) = d\rho_*(\omega) + \rho_*(\omega) \wedge \rho_*(\omega)$$

In particular, flat connections induce flat connections.
These observations show that every vector bundle can be constructed as an associated bundle of some principal bundle, and the results derived for principal bundles also hold for vector bundles via the correspondence between principal and vector bundles.

Parallel Transport

The horizontal subspace interpretation of parallel transport was mentioned in Section 1.2.
When our bundle is a vector bundle and connections can be thought of as covariant derivatives, then it makes sense to talk about constant sections of the bundle. These constant sections give the interpretation of parallel transport in the language of covariant derivatives. We will need the following theorem in the rest of this thesis.

Theorem 1.31. *If the curvature of a connection vanishes, then the parallel transport along a curve γ only depends on the homotopy class of γ.*

1.5 Complex manifolds

Complex structure on a manifold

The definition of a complex manifold is analogous to that of a smooth manifold.

Definition 1.32. Let M be a topological manifold. A **complex chart** (U, ϕ) is an open set $U \subset M$ and a homeomorphism $\phi : U \to \phi(U) \subset \mathbb{C}^n$.
A **complex atlas** is a collection of complex charts (U_i, ϕ_i) such that $M = \bigcup U_i$.
A **complex structure** on the manifold M is a maximal complex atlas such that if two charts (U_i, ϕ_i) and (U_j, ϕ_j) overlap, i.e. $U_i \cap U_j \neq \varnothing$ then the transition functions

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

between the charts are biholomorphic (i.e. the inverse is also holomorphic).
A **complex manifold** is topological manifold equipped with a complex structure.

Smooth maps and diffeomorphisms are replaced by holomorphic and biholomorphic maps.

Definition 1.33. Let M and N be complex manifolds, (U_i, ϕ_i) the atlas for M and (V_j, ϕ_j) the atlas for N. A map $f : M \to N$ between them is said to be **holomorphic** if $\phi_j \circ f \circ \phi_i^{-1}$ is holomorphic whenever $f(U_i) \cap V_j \neq \varnothing$. If $\phi_j \circ f \circ \phi_i^{-1}$ are biholomorphisms, then f is said to be a **biholomorhism**.

The analogue of a vector bundle is also straightforward.

Definition 1.34. A vector bundle is $\pi : E \to M$ is said to be a **holomorphic vector bundle** of rank n if the fibers are $\mathbb{C}^n$, E and M are complex manifolds, π is holomorphic, and the local trivialisations are biholomorphisms.

Example 1.35 (Holomorphic tangent bundle)**.** Let M be complex n-manifold, and $p \in M$. The tangent space T_pM at p is defined the same way as in the real case. In local coordinates the derivations $\{\partial_{z_1}, ..., \partial_{z_n}\}$ form a basis. The tangent bundle TM is the disjoint union of tangent spaces, and the topology of TM is defined the same way as in Example 1.5. The transition functions of 1.6 are given by the complex Jacobian of $\phi_\alpha \circ \phi_\beta^{-1}$.

Almost complex manifolds

Let V be a real vector space and $J : V \to V$ a linear map such that $J^2 = -I$ (where I is the identity on V). Such a J is called a complex structure on V, because we can equip V with the structure of a complex vector space by

$$(\alpha + i\beta)v = \alpha v + \beta J v$$

for $\alpha, \beta \in \mathbb{R}, v \in V$.

Now consider $V \otimes_{\mathbb{R}} \mathbb{C}$, the complexification of V. We can extend J to a map on $V \otimes_{\mathbb{R}} \mathbb{C}$ by $\mathbb{C}$-linearity such that $J^2 = -I$ still holds. It follows that J has two eigenvalues, i and $-i$. Let $V^{1,0}$ and $V^{0,1}$ be the eigenspaces of i and and $-i$, respectively. We can write

$$V \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus V^{0,1}$$

Definition 1.36. Let M be a smooth manifold of dimension $2n$. Suppose $J : TM \to TM$ is a vector bundle isomorphism such that $J_p : T_pM \to T_pM$ is a complex structure on each fiber. Then J is called an **almost complex structure** on the smooth manifold M.

Every complex manifold structure induces an almost complex structure on its underlying smooth manifold. Let M be a complex manifold and M_0 its underlying differentiable manifold. If $\{\partial_{z_1}, ..., \partial_{z_n}\}$ is a basis for T_pM, then $\{\partial_{x_1}, \partial_{y_1} ..., \partial_{x_n}, \partial_{y_n}\}$ is a basis for T_pM_0 where $z_k = x_k + iy_k$. Then

$$J_p(\partial_{x_k}) = \partial_{y_k} \qquad\qquad J_p(\partial_{y_k}) = -\partial_{x_k}$$

is an almost complex structure on M_0.

The converse, however, is not true in general. Given an almost complex structure J, the Newlander-Nirenberg theorem gives a necessary and sufficient condition for the existence of a complex structure that induces J. To state the theorem, we need to introduce differential forms on almost complex manifolds.

Differential forms on almost complex manifolds

Let V be a real vector space with a complex structure J, let $V_c = V \otimes_{\mathbb{R}} \mathbb{C}$ denote its complexification. The exterior algebras of $V^{1,0}$ and $V^{0,1}$ are naturally injected into the exterior algebra of V_c

$$\bigwedge V^{1,0} \longrightarrow \bigwedge V_c \qquad\qquad \bigwedge V^{0,1} \longrightarrow \bigwedge V_c$$

Let $\bigwedge^{p,q} V$ be the subspace of V_c that is generated by elements of the form $\alpha \wedge \beta$ where $\alpha \in \bigwedge^p V^{1,0}$ and $\beta \in \bigwedge^q V^{0,1}$. We can write

$$\bigwedge V_c = \sum_{d=0}^{2n} \sum_{p+q=d} \bigwedge^{p,q} V$$

Now suppose (M_0, J) is an almost complex manifold. J restricts to a complex structure on every fiber. Let $TM_0^{1,0}$ be the bundle of $(+i)$-eigenspaces, $TM_0^{0,1}$ the bundle of $(-i)$-eigenspaces, $T^*M_0^{1,0}$

and $T^*M_0^{0,1}$ their duals. Again, we have $T^*M_0^{1,0} \oplus T^*M_0^{0,1} = T^*M_c$ and the injections

$$\bigwedge T^*M_0^{1,0} \longrightarrow \bigwedge T^*M_c \qquad\qquad \bigwedge T^*M_0^{0,1} \longrightarrow \bigwedge T^*M_c$$

Definition 1.37. The **complex-valued differential forms** of type (p, q) on M_0 are the sections of $\pi : \bigwedge^{p,q} T^*M_0 \to M_0$. The space of such forms is denoted by $\mathcal{A}^{p,q}$.

In general, the exterior derivative maps a form α of type (p, q) to a form

$$d\alpha = \sum_{r+s=p+q+1} \pi_{r,s} d\alpha$$

where $\pi_{r,s}$ is the projection to the subspace of the forms of type (r, s). We define the ∂ and $\overline{\partial}$ operators that map a form of type (p, q) to a form of type $(p + 1, q)$ and $(p, q + 1)$, respectively.

$$\partial = \pi_{p+1,q} \circ d \qquad\qquad \overline{\partial} = \pi_{p,q+1} \circ d$$

Definition 1.38. If $d = \partial + \overline{\partial}$, then we say that the almost complex structure is integrable.

Now we can state the Newlander-Nirenberg theorem.

Theorem 1.39 (Newlander-Nirenberg)**.** *Let (X, J) be an almost complex manifold. There exists a unique complex manifold structure on X inducing J if and only if J is integrable.*

There is a similarity between this theorem and the theorem of Frobenius, as they both characterise the existence of manifold structures.

1.6 The theorems of de Rham and Hodge

Let X be a topological space. Let $\Delta_p = [e_0\, e_1 \ldots e_p] \subset \mathbb{R}^p$ be the standard p-simplex. A continuous map $\sigma : \Delta_p \to X$ is called a **singular** p**-simplex** in X. The **singular chain group** of degree p, denoted by $C_p(X)$, is the free abelian group generated by all singular p-simplices. The elements of $C_p(X)$ are called singular p-**chains**.
For $i = 0, \ldots, p$ we define the face map $F_{i,p} : \Delta_{p-1} \to \Delta_p$ to be the unique affine map that sends

$$e_0 \to e_0, \quad e_1 \to e_1, \quad \ldots \quad e_{i-1} \to e_{i-1}, \quad e_i \to e_{i+1}, \quad \ldots, \quad e_{p-1} \to e_p$$

The **boundary** of a p-simplex $\sigma : \Delta_p \to X$ is the $(p - 1)$-chain

$$\partial\sigma = \sum_{i=0}^{p} (-1)^i \sigma \circ F_{i,p}$$

A p-chain c is called a **cycle** if $\partial c = 0 \in C_{p-1}(X)$. Let $Z_p(X)$ denote the singular p-**cycles**. A p-chain b is called a **boundary** if there exists $c \in C_{p+1}(X)$ such that $\partial c = b$. Let $B_p(X)$ denote the singular p-**boundaries**. The boundary map satisfies $\partial^2 : C_p(X) \to C_{p-2}(X) = 0$, therefore $B_p(X) \subset Z_p(X)$.

Definition 1.40. The p-th **singular homology group** is defined as

$$H_{p,sing}(X) = H_p(C_\bullet, \partial) = \frac{Z_p(X)}{B_p(X)}$$

Remark 1.41. If R is a unital ring, we can take the singular p-simplices to be the generators of a free R-module. Let $H_{p,sing}(X, R)$ denote the homology groups we obtain using the same construction as above. If we take $R = \mathbb{Z}$, we get back $H_{p,sing}(X, \mathbb{Z}) = H_{p,sing}(X)$.

Let $(C_\bullet, \partial)$ be a graded chain complex (i.e. $\partial(C_i) \subset C_{i-1}$ and $\partial^2 = 0$), and G an abelian group. The dual chain complex (or cochain complex) is defined as

$$C^i = \mathrm{Hom}(C_i, G) = \{f : C_i \to G | \, f \text{ is a homomorphism}\}$$

with coboundary $\mathrm{d} : C^i \to C^{i+1}$ defined as

$$[\mathrm{d}(f)](c) = f(\partial c)$$

for $f \in C^i, c \in C_{i+1}$. The **singular cochains** of degree p of a topological space are defined as

$$C^i_{sing}(X) = \mathrm{Hom}(C_{i,sing}(X), G)$$

A p-cochain f is called a **cocycle** if $\mathrm{d}\, f = 0 \in C^{p+1}(X)$. Let $Z^p(X)$ denote the singular p-**cocycles**. A p-cochain g is called a **coboundary** if there exists $h \in C_{p-1}(X)$ such that $dh = g$. Let $B^p(X)$ denote the singular p-**coboundaries**.

Definition 1.42. The p-th **singular cohomology group** is defined as

$$H^p_{sing}(X) = H_p(C^\bullet, \mathrm{d}) = \frac{Z^p(X)}{B^p(X)}$$

So far these definitions work for any topological space. If in addition M has a smooth manifold structure, then we can replace the singular simplices (continuous maps $\Delta_p \to M$) with smooth simplices (smooth maps $\Delta_p \to M$). Following the exact same steps as above, we obtain the **smooth homology groups** $H_{p,\infty}(M)$. Also, the inclusion map $\iota : C_{p,\infty}(M) \to C_{p,sing}(M)$ commutes with the boundary map, so it induces a map $\iota_* : H_{p,\infty}(M) \to H_{p,sing}(M)$

Theorem 1.43. *For a smooth manifold M, the map $\iota_* : H_{p,\infty}(M) \to H_{p,sing}(M)$ is an isomorphism.*

Proof. [LeeSM], Theorem 18.7 □

The reason we need this theorem is that we want to pull back forms along simplices (maps $\Delta_p \to M$), and that is only possible if the simplices are smooth. Theorem 1.43 states that we don't lose anything if we only consider the smooth simplices.
Let ω be a closed p-form on some smooth manifold M, σ a smooth p-simplex in M. Define the integral of ω on σ as the integral of the pullback form in $\mathbb{R}^p$,

$$\int_\sigma \omega = \int_{\Delta_p} \sigma^* \omega$$

We extend this definition $\mathbb{R}$-linearly to $H_{p,sing}(M, \mathbb{R})$. Now we are ready to state the the de Rham theorem.

Theorem 1.44 (de Rham)**.** *Let M be a smooth manifold. The inclusion $\mathcal{I} : H^p_{dR}(M) \to H^p_{sing}(M, \mathbb{R})$ defined by*

$$[\mathcal{I}(\omega)](c) = \int_c \omega$$

is an isomorphism.

The formula is understood as follows: ω is a closed p-form representing the cohomology class $[\omega] \in H^p_{dR}(M)$ and c is a (smooth) singular p-chain representing the homology class $[c] \in H_{p,sing}(M) \cong H_{p,\infty}(M)$.

Proof. [LeeSM], Theorem 18.14 □

The Hodge Decomposition

Let (M, g) be a Riemannian manifold. The scalar product on the fibers of the tangent bundle can be extended to a scalar product on cotangent bundle by the isomorphism coming from the Riemannian structure $v \leftrightarrow g(., v)$. There is also a natural construction on the tensor products of these spaces that yields an inner product $\langle , \rangle$ on the space of differential forms over (M, g),

Definition 1.45. The **Hodge star operator** $\star : \mathcal{A}^k \to \mathcal{A}^{n-k}$ is defined as

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle dV_g$$

for $\alpha \in \mathcal{A}^k, \beta \in \mathcal{A}^{n-k}$.

Definition 1.46. The **Hodge inner product** $(,)$ of two forms α, β of the same degree is defined by

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle dV_g = \int_M \alpha \wedge \star\beta$$

where $d\mu$ is the volume form of g.

Definition 1.47. The **codifferential** $\delta : \mathcal{A}^k(M) \to \mathcal{A}^{k-1}(M)$ is

$$\delta = (-1)^{nk+1} \star d\star$$

We can now define the **Laplace-de Rham operator** $\Delta : \mathcal{A}^n \to \mathcal{A}^n$,

$$\Delta = d\delta + \delta d$$

Definition 1.48. A form ω is said to be **harmonic** if $\Delta\omega = 0$. When $\omega = f$ is a smooth function, this definition reduces to $\delta df = 0$. The space of harmonic forms of degree k is denoted by $\mathcal{H}^k$.

If M has empty boundary, then δ is the formal adjoint of the exterior derivative in the sense that

$$(d\alpha, \beta) = (\alpha, \delta\beta)$$

where $\alpha \in \mathcal{A}^k(M), \beta \in \mathcal{A}^{k+1}$. This follows from the theorem of Stokes,

$$0 = \int_M d(\alpha \wedge \star\beta) = \int_M d\alpha \wedge \star\beta - \alpha \wedge (-1)^{k+1} d \star \beta =$$
$$= \int_M d\alpha \wedge \star\beta - \alpha \wedge (-1)^{k(n-k)} \star (-1)^{k+1} \star d \star \beta = (d\alpha, \beta) - (\alpha, \delta\beta)$$

This also implies that on a manifold without boundary ω is harmonic if and only if $d\omega = 0$ and $\delta\omega = 0$.

Theorem 1.49 (Hodge). *Let M be a closed Riemannian manifold. Then*

$$\mathcal{A}^k = \mathcal{H}^k \oplus d\mathcal{A}^{k-1} \oplus \delta\mathcal{A}^{k+1}$$

i.e. every form ω of order k can be uniquely decomposed as

$$\omega = \omega_h + d\alpha + \delta\beta$$

where ω_h is a harmonic form, α is a $(k-1)$-form and β is a $(k+1)$-form.

Corollary 1.50. *For a closed form ω, Theorem 1.49 reduces to $\omega = \omega_h + d\alpha$.*

Definition 1.51. On a complex manifold the space of harmonic (p, q) forms are denoted by $\mathcal{H}^{p,q}$. In particular, **holomorphic** and **antiholomorphic** p-forms are defined as $\mathcal{H}^{p,0}$ and $\mathcal{H}^{0,p}$, respectively.

Theorem 1.52. *If M is a Riemann surface (complex 1-manifold) then*

$$\mathcal{H}^1 = \mathcal{H}^{1,0} \oplus \mathcal{H}^{0,1}$$

$$\mathcal{A}^{0,1} = \mathcal{H}^{0,1} \oplus \overline{\partial}\mathcal{A}^0$$

An other simple consequence of the Hodge decomposition theorem is that every de Rham cohomology class has a unique harmonic representative.

Theorem 1.53. *Let M be a closed Riemannian manifold. Then $H^k_{dR} \cong \mathcal{H}^k(M)$.*

1.7 Sheaf theory

Presheaves and Sheaves

Definition 1.54. Let X be a topological space, $\mathcal{C}$ a category. A **presheaf** $\mathscr{F}$ on X is

- A map $\mathscr{F}$ from the open sets of X into $\mathrm{ob}(\mathcal{C})$.
- A collection of restriction morphisms $r^U_V : \mathscr{F}(U) \to \mathscr{F}(V)$ for each $V \subset U$ satisfying
  - $r^U_U : id_{\mathscr{F}(U)}$
  - $r^U_W = r^V_W \circ r^U_V$, for every $W \subset V \subset U$

  This essentially defines a contravariant functor from the category of open sets of X into $\mathcal{C}$.

Definition 1.55. A presheaf $\mathscr{F}$ on X is said to be a **sheaf** if for every collection U_i of open subsets of X with $U = \cup U_i$ then $\mathscr{F}$ satisfies the following gluing axioms

$(S1)$ If $s_i \in \mathscr{F}(U_i)$ and if for $U_i \cap U_j \neq \varnothing$ we have

$$r^{U_i}_{U_i \cap U_j}(s_i) = r^{U_j}_{U_i \cap U_j}(s_j)$$

for all i, then there exists an $s \in \mathscr{F}(U)$ such that $r^U_{U_i}(s) = s_i$.

$(S2)$ If $s_1, s_2 \in \mathscr{F}(U)$ and $r^U_{U_i}(s_1) = r^U_{U_i}(s_2)$ for all i, then $s_1 = s_2$

Informally, $(S1)$ guarantees that the existence and $(S2)$ the uniqueness of the gluing.

Example 1.56. Let X and Y be topological spaces, and let $\mathscr{C}_{X,Y}$ be a presheaf over X defined as

- $\mathscr{C}_{X,Y}(U) = \{f : U \to Y \mid \text{f is continuous }\}$
- If $f \in \mathscr{C}_{X,Y}(U)$, then $r^U_V = f\big|_V$

In addition, $\mathscr{C}_{X,Y}$ satisfies $(S1)$ and $(S2)$, therefore this presheaf is actually a sheaf.

Definition 1.57. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves (resp. presheaves) over the same space X. A **morphism of sheaves**(resp. presheaves) is a collection of morphisms

$$h_U : \mathscr{F}(U) \to \mathscr{G}(U)$$

for each open set U such that $(r^U_V)_{\mathscr{G}} \circ h_U = h_V \circ (r^U_V)_{\mathscr{F}}$ for all $V \subset U \subset X$, where $(r^U_V)_{\mathscr{F}}$ and $(r^U_V)_{\mathscr{G}}$ are restriction morphisms of $\mathscr{F}$ and $\mathscr{G}$, respectively.
If the morphisms h_U are inclusions then $\mathscr{F}$ is said to be a **subsheaf** (resp. subpresheaf) of $\mathscr{G}$.
If the morphisms h_U are isomorphisms, then $\mathscr{F}$ and $\mathscr{G}$ are said to be **isomorphic sheaves** (resp. presheaves).

Definition 1.58. Let $(I, \leq)$ be a partially ordered set (i.e. $\leq$ is transitive and reflexive). If for every $i, j \in I$ there exists a $k \in I$ such that $i \leq k, j \leq k$, then $(I, \leq)$ is said to be a **directed set**.

Let $(I, \leq)$ be a directed set and $\mathcal{C}$ a category. A **direct system** is a collection of $A_i \in \text{ob}(\mathcal{C})$ for all i, and $\alpha_{i,j} : A_i \to A_j$ are morphisms for each $i \leq j$.

Example 1.59. Let $\mathscr{F}$ be a presheaf over X. The order on the open sets of X is given by $U_i \leq U_j \Leftrightarrow U_j \subset U_i$. Then $A_i = \mathscr{F}(U_i)$ is indexed by the open sets of X, and $\alpha_{i,j} = r^{U_i}_{U_j}$.

Definition 1.60. An object $A \in \text{ob}(\mathcal{C})$ together with morphisms $\alpha_i : A_i \to A$ is said to be the **direct limit** of the above direct system if

- $\alpha_i = \alpha_j \circ \alpha_{i,j}$ for all $i \leq j$
- For any other object $A' \in \text{ob}(\mathcal{C})$ and morphisms satisfying $\alpha'_i = \alpha'_j \circ \alpha_{i,j}$ there exists a unique morphism $\alpha : A \to A'$ such that $\alpha \circ \alpha_i = \alpha'_i$. (universal property)

The direct limit is denoted as $\varinjlim A_i$.

The second condition guarantees that if the direct limit exists, then it is unique up to isomorphism. If $\mathcal{C}$ is the category of abelian groups, rings (commutative with identity) or modules, then the direct limit exists and it is given by the following construction.
Let $(\{A_i\}, \{\alpha_{ij}\})$ be a direct system. Consider the disjoint union $\bigcup_{a_i \in A_i}(a_i, A_i)$ of the elements of A_i and the define the equivalence relation $\sim$ by $(a_i, A_i) \sim (a_j, A_j)$ if there exists a $k \geq i, j$ such that $\alpha_{i,k}(a_i) = \alpha_{j,k}(a_j)$. Let $[(a_i, A_i)]$ be the equivalence class of (a_i, A_i). Then

$$A = \bigcup_{a_i \in A_i} (a_i, A_i)/\sim \qquad \alpha_i(a_i) = [(a_i, A_i)]$$

is the direct limit of the system. From this point on we only deal with sheaves and presheaves of abelian groups, rings and modules.

Definition 1.61. Given a presheaf $\mathscr{F}$ over X, let

$$\mathscr{F}_x = \varinjlim_{x \in U} \mathscr{F}(U)$$

where the direct limit is taken with respect to the restriction morphisms. We call $\mathscr{F}_x$ the **stalk** of $\mathscr{F}$ at x. For $s \in \mathscr{F}(U)$ its image in the stalk is called the **germ** of s at x.

Consider the space

$$\tilde{\mathscr{F}} = \bigcup_{x \in X} \mathscr{F}_x$$

with the following topology. Fix an open set $U \subset X$ and $s \in \mathscr{F}(U)$. Let s_x denote the germ of s at x. Then we take $\{s_x \mid x \in U\}$ to be open for all pairs (s, U) and let these open sets generate the topology of $\tilde{\mathscr{F}}$.

Theorem 1.62. *Let $\mathscr{F}$ be a presheaf. The sections of $\tilde{\mathscr{F}}$ over open sets U (i.e.continuous maps $s : U \to Y$ such that $\pi \circ s = id_U$) form a sheaf. Let us denote this sheaf with $\bar{\mathscr{F}}$. Moreover, if $\mathscr{F}$ is a sheaf to begin with, then $\mathscr{F}$ and $\bar{\mathscr{F}}$ are isomorphic sheaves.*

Remark 1.63. The restriction morphisms are the natural restrictions of sections, and the abelian group structure of sections over the same open set is defined pointwise.

Definition 1.64. The sheaf $\bar{\mathscr{F}}$ is called the **sheaf generated by** $\mathscr{F}$.

By the second statement of Theorem 1.62 we may think of a sheaf $\mathscr{F}$ as the family of sections of the space $\tilde{\mathscr{F}}$ (which is just the disjoint union of stalks for all $x \in X$).

Cohomology

Definition 1.65. Let $\mathscr{A}, \mathscr{B}, \mathscr{C}$ be sheaves over X, and $g : \mathscr{A} \to \mathscr{B}, h : \mathscr{B} \to \mathscr{C}$ sheaf morphisms. The sequence

$$\mathscr{A} \xrightarrow{g} \mathscr{B} \xrightarrow{h} \mathscr{C}$$

is said to be **exact** at $\mathscr{B}$ if the induced sequence

$$\mathscr{A}_x \xrightarrow{g_x} \mathscr{B}_x \xrightarrow{h_x} \mathscr{C}_x$$

is exact at $\mathscr{B}_x$ for all $x \in X$. Short and long exact sequences of sheaves are defined similarly.

Definition 1.66. Let $\mathscr{F}^i$ be a collection of sheaves indexed by the integers. A **differential complex** of sheafs is a sequence of sheafs

$$... \xrightarrow{\alpha_{i-2}} \mathscr{F}^{i-1} \xrightarrow{\alpha_{i-1}} \mathscr{F}^i \xrightarrow{\alpha_i} \mathscr{F}^{i+1} \xrightarrow{\alpha_{i+1}} ...$$

connected by sheaf morphisms α_i such that $\alpha_{i+1} \circ \alpha_i = 0$.
A **resolution** of a sheaf $\mathscr{F}$ is a long exact sequence of sheaves of the form

$$0 \to \mathscr{F} \xrightarrow{j} \mathscr{F}^0 \xrightarrow{d^0} \mathscr{F}^1 \xrightarrow{d^1} ... \to \mathscr{F}^q \xrightarrow{d^q} \mathscr{F}^{q+1} \xrightarrow{d^{q+1}} ...$$

where 0 is the constant 0 sheaf.

Definition 1.67. A sheaf $\mathscr{F}$ is called **flabby** if for every open subset U the restriction map r_U^X : $\mathscr{F}(X) \to \mathscr{F}(U)$ is onto, i.e. every section of $\mathscr{F}$ on U can be extended to a section on X.

Let $\mathscr{F}$ be a sheaf and $\mathscr{F}^{[0]}$ the sheaf of germs of sections that are not necessarily continuous (maps $s : X \to \tilde{\mathscr{F}}$ such that $\pi \circ s = id_X$). It is clear that $\mathscr{F}^{[0]}$ is flabby and there is an injection $j : \mathscr{F} \to \mathscr{F}^{[0]}$.
Now define $\mathscr{F}^{[q]}$ as $(\mathscr{F}^{[q-1]})^{[0]}$. This is the sheaf of germs of sections of the sheaf $\mathscr{F}^{[q-1]}$ over X. By induction, we can think of the stalk $\mathscr{F}_x^{[q]}$ as the equivalence classes of maps $f : X^{q+1} \to \tilde{\mathscr{F}}$ such that $f(x_0, ..., x_q) \in \mathscr{F}_{x_q}$, where the equivalence relation is given as follows.
Let $f, g : X^{q+1} \to \tilde{\mathscr{F}}$ be maps such that $f(x_0, ..., x_q), g(x_0, ..., x_q) \in \mathscr{F}_{x_q}$. If they coincide on a set of the form

$$x_0 \in V_0 \quad x_1 \in V_1(x_0) \quad ... \quad x_q \in V_q(x_0, ..., x_{q-1}), \qquad V_q \subset ... \subset V_1 \subset V_0$$

where V_i is a neighbourhood of x_i depending on $x_0, ..., x_{i-1}$, then f and g are equivalent in the stalks in V_q.
The differential $d^q : \mathscr{F}^{[q]} \to \mathscr{F}^{[q+1]}$ is defined by

$$df(x_0, ..., x_{q+1}) = \sum_{0 \leq j \leq q} (-1)^j f(x_0, ..., \hat{x_i}, ..., x_{q+1}) + (-1)^{q+1} f(x_0, ..., x_q)(x_{q+1})$$

where the last term is the image of x_{q+1} under the continuous section $f(x_0, ..., x_q) \in \mathscr{F}_{x_q}$.

Theorem 1.68 (Godement, 1957)**.** *The complex $(\mathscr{F}^{[\bullet]}, d)$ is a resolution of the sheaf $\mathscr{F}$, called the canonical flabby resolution of $\mathscr{F}$.*

Now we are ready to define the cohomology groups.

Definition 1.69. Let $\mathscr{F}$ be a sheaf of abelian groups. For an integer $q \in \mathbb{Z}$, the q-th cohomology group of X with values in $\mathscr{F}$ is

$$H^q(X, \mathscr{F}) = H^q(\mathscr{F}^{[\bullet]}(X), d) = ker(d^q)/Im(d^{q-1})$$

with the convention $\mathscr{F}^q = 0, d^q = 0, H^q(X, \mathscr{F}) = 0$, when $q \leq 0$.

Definition 1.70. A sheaf $\mathscr{F}$ over X is said to be **acyclic** on an open set U if $H^q(U, \mathscr{F}) = 0$ for $q \geq 1$.

Theorem 1.71. *Flabby sheaves are acyclic on all open subsets.*

Proof. [Dem], Theorem 4.4 □

In Definition 1.69 we defined the cohomology groups by the canonical flabby resolution. The following theorem states the interesting result that by choosing any resolution by acyclic sheaves, we obtain the same cohomology groups.

Theorem 1.72 (de Rham-Weil). *Let $\mathscr{F}$ be a sheaf over X and*

$$0 \to \mathscr{F} \xrightarrow{j} \mathscr{A}^0 \xrightarrow{d^0} \mathscr{A}^1 \xrightarrow{d^1} \ldots \to \mathscr{A}^q \xrightarrow{d^q} \mathscr{A}^{q+1} \xrightarrow{d^{q+1}} \ldots$$

a resolution of $\mathscr{F}$, such that $\mathscr{A}^q$ is acyclic on X for all q. Then

$$H^q(\mathscr{A}^\bullet(X), d) \cong H^q(X, \mathscr{F})$$

Proof. [Dem], Theorem 6.4 □

Let M be a smooth manifold of dimension n. Also, let $\mathscr{E}^i$ denote the space of differential forms of degree q on M. Consider the resolution

$$0 \to \mathbb{R} \xrightarrow{j} \mathscr{E}^0 \xrightarrow{d} \mathscr{E}^1 \xrightarrow{d} \ldots \xrightarrow{d} \mathscr{E}^q \xrightarrow{d} \ldots \xrightarrow{d} \mathscr{E}^n \xrightarrow{d} 0$$

where d is the exterior derivative. The de Rham cohomology groups H^q_{dR} are defined as

$$H^q_{dR}(X, \mathbb{R}) = H^q(\mathscr{E}^\bullet(X), d)$$

It can be shown that $\mathscr{E}^q$ is acyclic for all q, therefore (by the de Rham-Weil isomorphism)

$$H^q_{dR}(X, \mathbb{R}) \cong H^q(X, \mathbb{R})$$

To conclude this section we state, without proof, a few relevant results of Algebraic Topology.

Theorem 1.73 (Five-lemma). *Given a commutative diagram of abelian groups*

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "k"] \arrow[d, "\gamma"] & D \arrow[r, "l"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\
A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "l'"] & E'
\end{tikzcd}$$

if the rows are exact and $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then γ is also an isomorphism.

Theorem 1.74 (Snake lemma). *If we are given a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r] & 0
\end{tikzcd}$$

with exact rows, then there is an exact sequence

$$0 \to ker\alpha \to ker\beta \to ker\gamma \xrightarrow{\delta} coker\alpha \to coker\beta \to coker\gamma$$

Theorem 1.75 (Mayer-Vietoris Exact Sequence)**.** *Let X be a topological space, U, V open subsets such that $U \cup V = X$. Then there exist maps δ_p such that the following sequence, called the Mayer-Vietoris sequence is exact:*

$$\dots \xrightarrow{\delta_{p-1}} H^p(X) \xrightarrow{\rho_p} H^p(U) \oplus H^p(V) \xrightarrow{\Delta_p} H^p(U \cap V) \xrightarrow{\delta_p} H^{p+1}(X) \to \dots$$

where $\rho = r_U^X + r_U^X$ and $\Delta = r_{U\cap V}^U - r_{U\cap V}^V$

Chapter 2

Smooth Hodge theory on compact Riemann surfaces

This chapter is mainly based on [GoXi].

2.1 Equivalence of Betti and de Rham groupoids

Definition 2.1. A **groupoid** is a category in which every morphism is invertible.

Definition 2.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An **equivalence of categories** is a functor $F : \mathcal{C} \to \mathcal{D}$ if it is

- full, i.e. for $x, y \in \mathrm{Obj}(\mathcal{C})$, the map $\mathrm{Mor}(x, y) \to \mathrm{Mor}(F(x), F(y))$ is surjective
- faithful, i.e. for $x, y \in \mathrm{Obj}(\mathcal{C})$, the map $\mathrm{Mor}(x, y) \to \mathrm{Mor}(F(x), F(y))$ is injective
- surjective on isomorphism classes, i.e. the induced map $F_* : \mathrm{Iso}(\mathcal{C}) \to \mathrm{Iso}(\mathcal{D})$ is surjective

The following example shows that equivalent categories are not necessarily isomorphic.

Example 2.3. Let $\mathcal{C}$ contain the abelian group $\mathbb{Z}_2$ and its identity morphism , and let $\mathcal{D}$ contain two isomorphic copies of $\mathbb{Z}_2$ with the identity morphisms of both and the isomorphism between them. These categories are easily seen to be equivalent, however they are clearly not isomorphic.

In this section we construct the Betti and de Rham groupoids and the equivalence between them. In what follows, let G be either $\mathbb{C}^*$ or $U(1)$. Also, let Σ be a closed, oriented, smooth 2-manifold with fundamental group π.

The Betti groupoid

Definition 2.4. The **Betti groupoid** is the category $(\mathrm{Hom}(\pi, G), G)$, i.e. the objects are the representations $\pi \to G$, and the morphisms are conjugations by the elements of G.

The fundamental group of a closed, oriented, smooth 2-manifold is

$$\langle A_1, B_1, \ldots, A_k, B_k \mid [A_1, B_1] \ldots [A_k, B_k] = 1\rangle$$

where k denotes the genus of the surface. Now let $\rho \in \mathrm{Hom}(\pi, G)$. Since G is abelian, its action is trivial and the relation

$$[\rho(A_1), \rho(B_1)] \ldots [\rho(A_k), \rho(B_k)] = 1_G$$

is automatically satisfied, therefore

$$\mathrm{Hom}(\pi, G)/G \cong \mathrm{Hom}(\pi, G) \cong G^{2k}$$

Hence $\mathrm{Hom}(\pi, \mathbb{C}^*)$ can be equipped with a natural complex structure.

The de Rham groupoid

Let $\pi : E \to \Sigma$ be the trivial complex line bundle over Σ with a fixed a trivialisation τ_0.

Definition 2.5. A **gauge transformation** ξ of $\pi : E \to \Sigma$ is a smooth bundle automorphism. Since E is a line bundle, this definition reduces to a smooth map $g : \Sigma \to \mathbb{C}^*$. Let $\mathcal{G}_l(E)$ denote the group of gauge transformations of E. Similarly, the $U(1)$-gauge group $\mathcal{G}_u(E)$ is the subgroup of smooth $\Sigma \to U(1)$ maps.

A trivialising section $\tau_0 : \Sigma \to E$ determines a connection D_0 by

$$D_0(s) = D_0(f\tau_0) = df(.)\tau_0$$

Every connection on D can be written as $D = D_0 + \eta$ for some $\eta \in \mathcal{A}^1(\Sigma)$, and D acts as

$$D(s) = D(f\tau_0) = df(.)\tau_0 + \eta(.)f\tau_0$$

Now let $\xi \in \mathcal{G}_l(E)$ be a gauge transformation represented by $g : \Sigma \to \mathbb{C}^*$. Define its action on $D = D_0 + \eta$ by

$$\xi(D_0 + \eta) = D_0 + \eta - g^{-1}dg$$

Let $\tau_0, \tau_0^* = g\tau_0$ be trivialisations, $s = f\tau_0$ a section, and D, D^* the connections determined by τ_0 and τ_0^*, respectively. Also, let ξ be the gauge transformation that is represented by g. Then we have

$$(\xi D)\big[\xi(f\tau_0)\big] = (d + \eta - g^{-1}dg)(gf\tau_0) = [d(gf) + gf\eta - f dg]\tau_0 = (df + f\eta)(g\tau_0) = D^*(f\tau_0^*)$$

A simple computation shows that although the $\mathcal{G}_l(E)$-action alters the connection, the curvature of the connection is invariant.

$$d(\eta - g^{-1}dg) = d\eta - dg^{-1} \wedge dg = d\eta + g^{-2}dg \wedge dg = d\eta$$

Let $\mathcal{F}_l(E)$ denote the set of flat $\mathbb{C}^*$ connections on the E, i.e. those connections for which $d\eta$ vanishes.

Definition 2.6. The **de Rham groupoid** is defined as $(\mathcal{F}_l(E), \mathcal{G}_l(E))$.

Complex structure of the de Rham groupoid

The $\mathcal{G}_l(E)$ action can be decomposed as the action of the identity component $\mathcal{G}_l(E)^0$ and the action of the group of connected components $\pi_0(\mathcal{G}_l(E))$ on the quotient. If $g \in \mathcal{G}_l(E)^0$ then $g = \exp f$ for some $f \in \mathcal{A}^0(\Sigma)$ and its action is given by

$$D_0 + \eta \xrightarrow{g} D_0 + \eta - (\exp f)^{-1}d(\exp f) = D_0 + \eta - df$$

The quotient $\mathcal{F}_l(E)/(\mathcal{G}_l(E)^0)$ is then $\mathrm{H}^1(\Sigma, \mathbb{C})$.
The action of $\pi_0(\mathcal{G}_l(E))$ will correspond to the inclusion of $\mathrm{H}^1(\Sigma, \mathbb{Z})$ into $\mathrm{H}^1(\Sigma, \mathbb{C})$. To see this, we first observe that the fundamental group of G is $\mathbb{Z}$, thus every smooth map $g \in \mathrm{Map}(\Sigma, G)$ induces

a homomorphism

$$\pi_1(\Sigma) \to \mathbb{Z}$$

determining and element of

$$\mathrm{Hom}(\pi_1(\Sigma), \mathbb{Z}) \cong \mathrm{Hom}(\mathrm{Ab}(\pi_1(\Sigma)), \mathbb{Z}) = \mathrm{H}^1(\Sigma, \mathbb{Z})$$

The resulting group homomorphism $\mathrm{Map}(\Sigma, G) \to \mathrm{H}^1(\Sigma, \mathbb{Z})$ induces an isomorphism

$$\pi_0(\mathrm{Map}(\Sigma, G)) \cong \mathrm{H}^1(\Sigma, \mathbb{Z})$$

Thus $\mathcal{F}_l(E)/\mathcal{G}_l(E)$ can be identified with $\mathrm{H}^1(\Sigma, \mathbb{C})/\,\mathrm{H}^1(\Sigma, \mathbb{Z})$ and inherits the complex structure J

$$J(\eta) = i\eta$$

The Equivalence

Holonomy assigns to each loop based at x_0 a homomorphism

$$hol_{x_0} : \pi_1(\Sigma, x_0) \to \mathbb{C}^*$$

If $\xi \in \mathcal{G}_l(E)$(corresponding to $g : \Sigma \to C^*$) is a gauge transformation, then the evaluation of g at x_0 is a $\mathcal{G}_l(E) \to \mathbb{C}^*$ homomorphism. These two maps together give the **holonomy functor**.

Theorem 2.7 (Riemann-Hilbert correspondence)**.** *The functor*

$$\mathrm{hol} : (\mathcal{F}_l(E), \mathcal{G}_l(E)) \to (\mathrm{Hom}(\pi, \mathbb{C}^*), \mathbb{C}^*)$$

is full, faithful, and surjective on isomorphism classes, ie. an equivalence of the Betti and de Rham groupoids.

2.2 Equivalence of Dolbeault and de Rham groupoids

The Dolbeault groupoid

Let $E \to \Sigma, \tau, D, D_0$ be as in the previous section. Then we can decompose D_0 as

$$D_0(f\tau) = df(.)\tau = \partial f(.)\tau + \bar{\partial} f(.)\tau = D_0'(f\tau) + D_0''(f\tau)$$

Definition 2.8. A **holomorphic structure** is an operator

$$D'' = D_0'' + \Psi \qquad\qquad \Psi \in \mathcal{A}^{0,1}(X)$$

that satisfies

$$f\tau \xrightarrow{D''} (\bar{\partial} f + f\Psi)\tau$$

Let $\mathrm{Hol}(E)$ denote the space of all holomorphic structures on E.

Definition 2.9. A **Higgs field** on a holomorphic vector bundle (E, D'') is a $(1, 0)$ form Φ on Σ taking values in the endomorphism bundle $\mathrm{End}(E)$ that is holomorphic with respect to the holomorphic structure of $T^*\Sigma \otimes \mathrm{End}(E)$. A **Higgs bundle** is the triple (E, D'', Φ) where (E, D'') is a holomorphic vector bundle and Φ is a Higgs field on (E, D'').

When E is a line bundle, a Higgs field is just a holomorphic 1-form,

$$\mathrm{Higgs}(E) = \mathrm{Hol}(E) \times \mathcal{H}^{1,0}(\Sigma)$$

The gauge group $\mathcal{G}_l(E)$ acts on $\mathrm{Hol}(E)$ by

$$D_0'' + \Psi \xrightarrow{\xi} D_0'' + \Psi - g^{-1}\bar{\partial} g$$

and on Higgs fields by conjugation. Since $\mathrm{End}(E)$ is commutative, the action of $\mathcal{G}_l(E)$ on $\mathcal{H}^{1,0}(\Sigma)$ is trivial. Denote the resulting groupoid by $(\mathrm{Higgs}(E), \mathcal{G}_l(E))$.

Complex structure of the Dolbeault groupoid

The $\mathcal{G}_l(E)$ action on $\mathrm{Hol}(E)$ can be decomposed as the action of the identity component $\mathcal{G}_l(E)^0$ and the action of the group of connected components $\pi_0(\mathcal{G}_l(E))$ on the quotient. If $g \in \mathcal{G}_l(E)^0$ then $g = \exp f$ for some $f \in \mathcal{A}^0$ and its action is given by

$$D_0'' + \Psi \xrightarrow{g} D_0'' + \Psi - \overline{\partial} f$$

The quotient $\mathrm{Hol}(E)/(\mathcal{G}_l(E)^0)$ can be identified with $\mathcal{A}^{0,1}/\overline{\partial}\mathcal{A}^0 = \mathcal{H}^{0,1}$ by the Hodge decomposition theorem. Let $V = \mathcal{H}^{0,1}$ denote the space of antiholomorphic 1-forms.
The action of $\pi_0(\mathcal{G}_l(E)) \cong \mathrm{H}^1(\Sigma, \mathbb{Z})$ corresponds to the action of $\mathrm{H}^1(\Sigma, \mathbb{Z})$ on $\mathcal{H}^{0,1}$ by translation via the map

$$\mathrm{H}^1(\Sigma, \mathbb{Z}) \xrightarrow{i} \mathrm{H}^1(\Sigma, \mathbb{C}) \xrightarrow{p} \mathcal{H}^{0,1} = V$$

where i is the natural inclusion of $\mathrm{H}^1(\Sigma, \mathbb{Z})$ into $\mathrm{H}^1(\Sigma, \mathbb{C})$ and p is the projection to the subspace of antiholomorphic 1-forms. The image of $\mathrm{H}^1(\Sigma, \mathbb{Z})$ under $p \circ i$ is a lattice $L \subset V$ of rank $\dim_{\mathbb{R}}(V)$. The quotient V/L is then a torus of dimension $\dim_{\mathbb{R}}(V)$, the **Jacobian** of Σ, denoted by $\mathrm{Jac}(\Sigma)$.
The Dolbeault isomorphism identifies the space of antiholomorphic 1-forms and the first cohomology group of the sheaf of germs of holomorphic functions,

$$\mathcal{H}^{0,1}(\Sigma) \cong \mathrm{H}^1(\mathcal{O}_\Sigma, \Sigma)$$

To summarise, $\mathrm{Higgs}(E)/\mathcal{G}_l(E)$ identifies with

$$\mathrm{Jac}(\Sigma) \times \mathcal{H}^{1,0} = V/L \times \overline{V}$$

The complex structure I of $V/L \times \overline{V}$ arises from those of V and $\overline{V}$. Namely, for $\Psi \in V$ and $\Phi \in \overline{V}$

$$(\Psi, \Phi) \xrightarrow{I} (i\Psi, i\Phi)$$

The Equivalence

Definition 2.10. Let $\mathrm{Her}(E)$ denote the space of **Hermitian metrics**, i.e. tensors of type $TM^2 \to \mathbb{C}$ that restrict to positive definite Hermitian forms $\langle , \rangle_H$ on each fiber.

In a trivialisation$\{s_1, ..., s_n\}$ the Hermitian metric H can be represented by a positive definite Hermitian matrix h_{ij}. When E is a complex line bundle and τ_0 is a trivialisation, the matrix h_{ij} reduces to positive real function h

$$\langle s_1, s_2 \rangle_H = \langle f_1\tau_0, f_2\tau_0 \rangle_H = f_1 h \overline{f_2}$$

Gauge transformations act on Hermitian metrics by

$$\langle s_1, s_2 \rangle_H \xrightarrow{\xi} \langle g^{-1}s_1, g^{-1}s_2 \rangle_H = \langle s_1, s_2 \rangle_{\xi H}$$

i.e. we have the identity

$$\langle \xi(s_1), \xi(s_2) \rangle_{\xi H} = \langle s_1, s_2 \rangle_H$$

and the functions representing H and (ξH) are related by

$$h \xrightarrow{\xi} g^{-1}\overline{g^{-1}}h = |g|^{-2}h$$

Definition 2.11. A connection D is said to be **unitary** with respect to H if it satisifies

$$d\langle s_1, s_2\rangle_H = \langle Ds_1, s_2\rangle_H + \langle s_1, Ds_2\rangle_H$$

for any two sections s_1, s_2. Equivalently, we say that H is **parallel** with respect to D.

Let $s_i = f_i\tau_0$ The above equation then becomes

$$d(f_1 h\overline{f_2}) = (df_1 + f_1\eta)h\overline{f_2} + f_1 h\overline{d(df_2 + f_2\eta)}$$

which simplifies to

$$f_1 dh\overline{f_2} = f_1\eta h\overline{f_2} + f_1 h\overline{f_2\eta} \quad \Leftrightarrow \quad dh = h(\eta + \overline{\eta})$$

which shows that D is unitary with respect to H if and only if

$$h^{-1}dh = 2Re(\eta)$$

Let $p : \tilde{\Sigma} \to \Sigma$ be the universal cover of Σ. Also, let p^*E, p^*D, p^*H and $p^*\tau_0$ be the pullbacks of $E \to \Sigma, D, H$ and τ_0, respectively. We want to find a p^*D-parallel trivialisation τ of p^*E.
We look for τ in the form

$$\tau = \varphi(p^*\tau_0) \tag{2.1}$$

where φ is a smooth function on $\tilde{\Sigma}$. Such a trivialisation needs to satisfy

$$0 = p^*D(\tau) = p^*D(\varphi(p^*\tau_0)) = d\varphi(.)p^*\tau_0 + p^*\eta(.)\varphi(p^*\tau_0)$$

therefore

$$p^*\eta = -\varphi^{-1}d\varphi = -d\log\varphi \tag{2.2}$$

Since $\tilde{\Sigma}$ is simply connected, $\mathrm{H}_1(\tilde{\Sigma}) = 0$ which also implies $\mathrm{H}^1(\tilde{\Sigma}) = 0$, i.e. all closed forms are exact, and the above equation can be solved. Thus we can define $\varphi_D : \tilde{\Sigma} \to \mathbb{C}$ in a way that makes $\varphi_D(p^*\tau_0)$ parallel with respect to p^*D.
Now we define a metric $\tilde{H}$ on $\tilde{\Sigma}$ which is, in the trivialisation $p^*\tau_0$, represented by

$$\tilde{h} = |\varphi_D|^2 p^*h$$

Equivalently, in the p^*D-parallel trivialisation $\tau = |\varphi_D|^2 p^*\tau_0$ it is represented by

$$|\varphi_D|^{-2}\tilde{h} = p^*h$$

Since φ_D is equivariant with respect to the holonomy group of the connection, we can define $p_*\varphi_D$ locally on Σ.

Definition 2.12. Define the metric H_0 by $\langle \tau_0, \tau_0\rangle_{H_0} = 1$, i.e. H_0 is represented (with respect to the trivialisation τ_0) by $h_0 = 1$. The subset of $\mathcal{F}_l(E)$ containing connections that are unitary with respect to H_0 will be denoted by $\mathcal{F}_u(E)$.

As D_0 is determined by the trivialisation τ_0, so is H_0. In other words, once we pick a trivisalisation τ_0, we automatically obtain the connection D_0 and the metric H_0.

Definition 2.13. Let H be a Hermitian metric, D'' a holomorphic structure. The connection D is said to be **compatible** with D'' and H if

- the $(0,1)$ part of D equals D'' and
- D is unitary with respect to H

The connection D is also called the **Chern-connection**.

For any pair of D'' and H exists a unique connection satisfying these conditions. Namely, if $D'' = D_0'' + \mu$, and H is represented by h, then D is given by

$$D_0 + \mu - \overline{\mu} + h^{-1}\partial h$$

Definition 2.14. A Hermitian metric H is said to be the **Hermitian-Einstein** metric with respect to D'' if D (the connection compatible with D'' and H) is flat.

Let $\mathrm{Hol}_u(E)$ denote the subset of $\mathrm{Hol}(E) \times \mathrm{Her}(E)$ consisting of (D'', H) such that H is Hermitian-Einstein with respect to D''.

Theorem 2.15. *There is an equivalence of groupoids $\mathcal{T}$*

$$(\mathcal{F}_u(E), \mathcal{G}_u(E)) \xrightarrow{\mathcal{T}} (\mathrm{Hol}_u(E), \mathcal{G}_l(E))$$

where $\mathcal{T}$ acts on the objects by

$$D \xrightarrow{\mathcal{T}} (D^{(0,1)}, H_0)$$

and on morphisms by the inclusion $\mathcal{G}_u(E) \hookrightarrow \mathcal{G}_l(E)$.

Theorem 2.16. *Let $D'' \in \mathrm{Hol}(E)$. Then there exists a Hermitian metric represented by h such that $(D'', h) \in \mathrm{Hol}_u(E)$.*

Proof. Let D'' be a holomorphic structure. By Theorem 1.52 $D'' = D_0'' + \Psi_0 + \overline{\partial} f$, where Ψ_0 is antiholomorphic and $f \in \mathcal{A}^0$. The connection $D = D_0 + \Psi_0 - \overline{\Psi}_0 + df$ is flat ($\eta = \Psi_0 - \overline{\Psi}_0 + df$ is closed) and $D^{0,1} = D''$. We just need to find a metric that is parallel with respect to D. We define H by the equation $\eta + \overline{\eta} = d \log h$ to obtain a desired metric that is defined up to multiplication by a positive constant ($h = c\, e^{2Ref}$ for any $c \in \mathbb{R}^+$). ☐

Note that Theorem 2.15 is about unitary connections and unitary gauge-transformations, whereas the de Rham groupoid is $(\mathcal{F}_l(E), \mathcal{G}_l(E))$. In the rest of this section we extend the unitary case to the linear one, and relate the de Rham groupoid to the Dolbeault groupoid.

Let $E \to \Sigma$ be a trivial line bundle, D a flat connection, H a Hermitian metric, τ_0 a trivialisation. With respect to τ_0 we can write $D = D_0 + \eta$ for some $\eta \in Z^1(\Sigma)$. Let ϕ and ψ be the real and imaginary parts of η. Since ϕ and ψ are purely real and imaginary, η is closed if and only if ϕ and ψ are closed. We can then form the unitary part of D

$$D_H = D_0 + \psi + \frac{1}{2} h^{-1} dh$$

Let ϕ_1 denote $D - D_H$, i.e. $\phi_1 = \phi - \frac{1}{2} h^{-1} dh$.

Definition 2.17. A Hermitian metric H is said to be **harmonic** with respect to D if $\log \tilde{h}$ is a harmonic function on $\tilde{\Sigma}$.

Theorem 2.18. *A Hermitian metric H is harmonic if and only if ϕ_1 is a harmonic 1-form.*

Proof. By definition H is harmonic if $\log \tilde{h}$ is harmonic on $\tilde{\Sigma}$. Since $\log \tilde{h}$ is a 0-form, we automatically have $d\delta \log \tilde{h} = 0$. Thus $\log \tilde{h}$ is harmonic if and only if

$$0 = \delta d \log \tilde{h} \qquad \text{by definition, } \tilde{h} = |\varphi_D|^2 p^* h$$

$$
\begin{aligned}
&= \delta d(\log\varphi + \log\overline{\varphi} + \log p^*h) && \text{by (2.2)}\\
&= -\,\delta(p^*\eta + p^*\overline{\eta} - d\log p^*h) && \text{by definition, } \phi = Re(\eta)\\
&= -\,2\delta(p^*\phi - \frac{1}{2}d\log p^*h) && \text{by definition, } \phi_1 = \phi - \frac{1}{2}d\log h\\
&= -\,2\delta p^*\phi_1\\
&= -\,2p^*(\delta\phi_1)
\end{aligned}
$$

Since $d\phi_1 = 0$, H is a harmonic metric if and only if ϕ_1 is a harmonic 1-form on Σ. □

Theorem 2.19. *Let D be flat. Then there exists a Hermitian metric that is harmonic with respect to D.*

Proof. Let $D = D_0 + \phi + \psi$ as above, where ψ is a real-valued 1-form. By the Hodge decomposition we have

$$\phi = \phi_h + df$$

where ϕ_h is harmonic with respect to the metric H_0. Setting $\phi_1 = \phi_h$ and $df = \frac{1}{2}h^{-1}dh$ we obtain a metric represented by $h = e^{2f}$ that is harmonic by Theorem 2.18 and unique up to multiplication by a positive constant. □

Now let D be a flat connection, H its harmonic metric (as in Theorem 2.19) and put $D = D_H + \phi_1$. Also, let $\Phi = \phi_1^{(1,0)}$. Since ϕ_1 is harmonic, $\overline{\partial}\Phi = 0$.

Theorem 2.20. *The resulting functor $\mathcal{S}$*

$$(\mathcal{F}_l(E), \mathcal{G}_l(E)) \xrightarrow{\ \mathcal{S}\ } (\mathrm{Higgs}(E), \mathcal{G}_l(E))$$

that acts on objects by

$$D \xrightarrow{\mathcal{S}} (D_H^{(0,1)}, \Phi)$$

is an equivalence of groupoids.

Proof. First we prove surjectivity on isomorphism classes. Let (D'', Φ) be a Higgs pair. Suppose $D'' = D_0'' + \Psi$. By Theorem 2.16 there exists a metric H that is Hermitian-Einstein with respect to D''. Let D_H denote the Chern connection (Definition 2.13). Then $D = D_H + \Phi + \overline{\Phi}$ is flat, and $\mathcal{S}(D) = (D'', \Phi)$.

Next we show that $\mathcal{S}$ is faithful and full. Let D_1, D_2 be flat connections. If they are $\mathcal{G}_l(E)$-equivalent, then there exists $g_0 : \Sigma \to \mathbb{C}^*$ such that $D_1 = D_2 + g_0^{-1}dg_0$. For any other g_1 that satisfies $D_1 = D_2 + g_1^{-1}dg_1$, we have $g_0^{-1}dg_0 = g_1^{-1}dg_1$, that implies $g_0 = cg_1$ for some constant $c \in \mathbb{C}^*$, and $\mathrm{Mor}(D_1, D_2) \cong \mathbb{C}^*$. If D_1 and D_2 are not $\mathcal{G}_l(E)$-equivalent, then $\mathrm{Mor}(D_1, D_2) = \varnothing$. Similarly, if $(D'', \Phi)_1$ and $(D'', \Phi)_2$ are $\mathcal{G}_l(E)$-equivalent, then $\mathrm{Mor}((D'', \Phi)_1, (D'', \Phi)_2) \cong \mathbb{C}^*$, otherwise $\mathrm{Mor}(D_1, D_2) = \varnothing$.

It remains to show that $\mathcal{S}(D_1)$ is equivalent to $\mathcal{S}(D_2)$ if and only if D_1 is equivalent to D_2. Let $D = D_0 + \eta$ and ξD be equivalent connections. Then

$$D_H = D_0 + iIm(\eta) + \frac{1}{2}h^{-1}dh$$

$$
\begin{aligned}
(\xi D)_{\xi H} &= D_0 + iIm(\eta + g^{-1}dg) + \frac{1}{2}|g|^2h^{-1}d|g|^{-2}h\\
&= D_0 + iIm(\eta + g^{-1}dg) + \frac{1}{2}d\log(|g|^{-2}h)
\end{aligned}
$$

Also,

$$(\xi D)^{0,1} = \xi(D^{0,1})$$

Therefore the holomorphic structures of $\mathcal{S}(D)$ and $\mathcal{S}(\xi D)$ are related by ξ. Moreover,

$$D - D_H = Re(\eta) - \frac{1}{2}h^{-1}dh$$

We also have

$$\begin{aligned}
\xi D - (\xi D)_{\xi H} &= Re(\eta - g^{-1}dg) - \frac{1}{2}|g|^2 h^{-1} d|g|^{-2} h \\
&= Re(\eta - d\log g) - \frac{1}{2} d\log(|g|^{-2}h) \\
&= Re(\eta - d\log g) - \frac{1}{2}(d\log g^{-1} + d\log \overline{g^{-1}} - d\log h) \\
&= Re(\eta - d\log g) - \frac{1}{2}(-d\log g - \overline{d\log g} - d\log h) \\
&= Re(\eta) - \frac{1}{2} d\log h
\end{aligned}$$

i.e. the Higgs fields of $\mathcal{S}(D)$ and $\mathcal{S}(\xi D)$ agree. Conversely, if $\mathcal{S}(D_1)$ and $\mathcal{S}(D_2)$ are ξ-related, then so are D_1 and D_2. This completes the proof. □

Chapter 3

Logarithmic theory on compact Riemann surfaces

Let Σ be a closed Riemann-surface, $p_1, p_2, ..., p_n \in \Sigma$ distinct, $\beta_1, \beta_2, ..., \beta_n \in \mathbb{R}$, $\mu_1, \mu_2, ..., \mu_n \in \mathbb{C}$. Let D be a flat connection on the trivial line bundle over $\Sigma \setminus \{p_1, ..., p_n\}$ such that τ_0 is a $D^{0,1}$-holomorphic trivialisation and for every p_i

$$D^{1,0} = \partial + \frac{\mu_i}{z_i} dz_i + O(1) dz_i \tag{3.1}$$

where z_i are local coordinates in a neighbourhood of p_i such that $z_i(p_i) = 0$.

Remark 3.1. A trivialisation τ_0 is $D^{0,1}$-holomorphic if $D^{0,1}(\tau_0) = 0$, i.e. if $D = D_0 + \eta$, then $\eta^{0,1} = 0$.

Theorem 3.2. *If $\beta_i = Re(\mu_i)$ for $i \in \{1, 2, ..., n\}$, then there exists a unique (up to multiplication by constants) metric H that is harmonic with respect to D, for which*

$$\lim_{z_i \to 0} \frac{\langle \tau_0, \tau_0 \rangle_H}{|z_i|^{2\beta_i}} = \alpha_i \neq 0 \tag{3.2}$$

i.e the above limit exists and it is non-zero.

Remark 3.3. Let η be a connection 1-form of a connection D that satisfies (3.1). If we substitute η into (2.2) we obtain the equation

$$-d \log \varphi_D = \frac{\mu_i}{z_i} dz_i + O(1) dz_i = \mu_i d \log z_i + df_0 = \mu_i d \log z_i + d \log f = d \log(z_i^{\mu_i} f)$$

where $df_0 = O(1) dz_i$ and $f = e^{f_0}$ is non-zero. Therefore

$$\log \varphi_D^{-1} = \log(z_i^{\mu_i} f) + c_0 \quad \Rightarrow \quad \varphi_D = c z_i^{-\mu_i} f^{-1}$$

where $c = e^{-c_0}$ is non-zero.

Also, let h represent a metric H that satisfies (3.2). Then for every p_i, with local coordinate z_i

$$\begin{aligned}
\lim_{z_i \to 0} \tilde{h}(z_i) &= \lim_{z_i \to 0} |\varphi_D(z_i)|^2 h(z_i) \\
&= \lim_{z_i \to 0} |c z_i^{-\mu_i} f(z_i)^{-1}|^2 (\alpha_i |z_i|^{2\beta_i}) \\
&= \lim_{z_i \to 0} (|z_i|^{-2Re\mu_i} |c|^2 |f(z_i)|^{-2})(\alpha_i |z_i|^{2\beta_i}) \\
&= \alpha_i |c|^2 |f(0)|^{-2}
\end{aligned}$$

which is non-zero by assumption.

Definition 3.4. The **weight of the associated local system** at p_i is defined as the constant $c \in \mathbb{R}$ for which $\langle \tau, \tau \rangle_H = \tilde{h}$ asymptotically equals $\lambda |z_i|^{2c}$ for some $\lambda \neq 0$ as $z_i \to 0$.

By the above computation, the weight of our associated local system is 0 at every p_i.
During the proof of Theorem 3.2 we are going to encounter a differential equation, which has — because of the singular points p_i — non-smooth terms. Before starting the proof, we need a couple of computations to understand the distributions (in the generalised functions sense) that will appear in the equation.

Proposition 3.5. *Let z be a local coordinate on a neighbourhood U of p that extends smoothly to a global function such that $z(\Sigma \setminus \{p\}) \in \mathbb{C}^*$. Then*

1. $\Delta \log(|z|) = 2\pi\delta_p + s_1$
2. $\star d \star Re(\mu \frac{dz}{z}) = -2\pi Re(\mu)\delta_p + s_2$

where δ_p is the Dirac delta distribution at p, and s_1, s_2 denote compactly supported smooth functions on $\Sigma \setminus \backslash\{p_1, ..., p_n\}$.

Proof. Let $D^2(\varepsilon)$ be the disk of radius ε, centered at p with boundary $S^1(\varepsilon)$ and f a test function on Σ. If f is supported on $\Sigma \setminus \{p\}$, then both $\Delta \log |z| f$ and $Re(\mu \frac{dz}{z})f$ are smooth. Now let $suppf \subset U$.

1. After some formal manipulations of the distributions, and by Stokes' theorem,

$$\begin{aligned}
\langle \Delta \log |z|, f\rangle &= \int_\Sigma \Delta(\log |z|) f dV_\Sigma = \int_\Sigma - \star d \star d(\log |z|) f dV_\Sigma = - \int_\Sigma d \star d(\log |z|) f \\
&= - \int_\Sigma \star d \log |z| \wedge df = - \int_U \star\Big(\tfrac{\partial \log |z|}{\partial r} dr + \tfrac{\partial \log |z|}{\partial \theta} d\theta\Big) \wedge \Big(\tfrac{\partial f}{\partial r} dr + \tfrac{\partial f}{\partial \theta} d\theta\Big) \\
&= - \int_U \tfrac{\partial \log |z|}{\partial r} r d\theta \wedge \tfrac{\partial f}{\partial r} dr = \int_U \tfrac{\partial f}{\partial r} dr \wedge d\theta = \lim_{\varepsilon \to 0} \int_{U \setminus D^2(\varepsilon)} d\big(f d\theta\big) \\
&= \lim_{\varepsilon \to 0} \int_{S^1(\varepsilon)} f d\theta = \lim_{\varepsilon \to 0} \int_{S^1(\varepsilon)} \frac{f}{r} dV_{S^1(\varepsilon)} = 2\pi f(p)
\end{aligned}$$

2. Similarly,

$$\begin{aligned}
\langle \star d \star Re(\mu \tfrac{dz}{z}), f\rangle &= \int_\Sigma \star d \star Re(\mu \tfrac{dz}{z}) f dV_\Sigma = \int_\Sigma d \star Re(\mu \tfrac{dz}{z}) f = \int_\Sigma \star Re(\mu \tfrac{dz}{z}) \wedge df \\
&= \int_U \star Re(\mu d \log z) \wedge df = \int_U \star Re\Big(\mu d(\log r + \log e^{i\theta})\Big) \wedge df \\
&= \int_U \star Re\Big(\mu(\tfrac{dr}{r} + i d\theta)\Big) \wedge df = \int_U \star\Big(Re(\mu)\tfrac{dr}{r} - Im(\mu) d\theta\Big) \wedge df \\
&= \int_U \Big(Re(\mu) d\theta + Im(\mu)\tfrac{dr}{r}\Big) \wedge df = \int_U \Big(Re(\mu) d\theta + Im(\mu)\tfrac{dr}{r}\Big) \wedge \Big(\tfrac{\partial f}{\partial r} dr + \tfrac{\partial f}{\partial \theta} d\theta\Big) \\
&= Re(\mu) \int_U \tfrac{\partial f}{\partial r} d\theta \wedge dr + Im(\mu) \int_U \tfrac{1}{r}\tfrac{\partial f}{\partial \theta} dr \wedge d\theta \\
&= - 2\pi Re(\mu) f(p) + Im(\mu) \int_U d\big(\tfrac{f}{r} dr\big) = -2\pi Re(\mu)\delta_p + Im(\mu) \lim_{\varepsilon \to 0} \int_{S^1(\varepsilon)} \tfrac{f}{r} dr \\
&= - 2\pi Re(\mu) f(p)
\end{aligned}$$

□

Proof of Theorem 3.2. Let H^* (represented by h^*) be a metric that satisfies (3.2). Such a metric can easily be constructed locally in a small neighbourhood around p_i and then the local definitions can be glued together by smooth bump functions. We will look for the desired metric in the form of $h = e^{2\rho}h^*$ for some $\rho : \Sigma \to \mathbb{R}$. Such a metric needs to satisfy (Theorem 2.18)

$$0 = \Delta\Big(Re(\eta) - \frac{1}{2}h^{-1}dh\Big) = \Delta\Big(Re(\eta) - e^{-\rho}de^{\rho} - \frac{dh^*}{2h^*}\Big)$$

Therefore

$$\begin{aligned} 0 &= \Delta\Big(Re(\eta) - d\rho - \frac{1}{2}d\log h^*\Big) \\ &= -d\star d\star\Big(Re(\eta) - d\rho - \frac{1}{2}d\log h^*\Big) \\ &= -d\Big(\star d\star Re(\eta) + \Delta\rho + \frac{1}{2}\Delta\log h^*\Big) \end{aligned}$$

which implies

$$\Delta\rho = -\star d\star Re(\eta) - \frac{1}{2}\Delta\log h^* + c \tag{3.3}$$

for some constant c. Then by Proposition 3.5,

$$\Delta\rho = S + c + 2\pi\sum_{i=1}^{n}\big(Re(\mu_i) - \beta_i\big)\delta_{p_i}$$

for some smooth function S on Σ. The Dirac distributions cancel each other by assumption, and (3.3) simplifies to

$$\Delta\rho = S + c$$

By Theorem 1.49 the 0-forms can be decomposed as

$$\mathcal{A}^0 = \mathcal{H}^0 \oplus \delta\mathcal{A}^1$$

On a closed manifold harmonic functions are just constants, therefore we can pick c such that the RHS of (3.3) is in the image of δ. It remains to prove that the image of $\delta : \mathcal{A}^1 \to \mathcal{A}^0$ is exactly the image of $\Delta : \mathcal{A}^0 \to \mathcal{A}^0$.

This follows from Theorem 1.49,

$$\delta(\mathcal{A}^1) = \delta(\mathcal{H}^1 \oplus d\mathcal{A}^0 \oplus \delta\mathcal{A}^2) = \delta d\mathcal{A}^0 = \Delta(\mathcal{A}^0)$$

This shows the existence of the function ρ that solves (3.3) and its uniqueness up to the addition of harmonic functions. Harmonic functions on compact manifolds are just constants, therefore ρ is unique up to an additive constant, which in turn defines H up to a multiplicative constant.

$\square$

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