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Merton's Portfolio Problem

MS Thesis

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2016
Budapest, Hungary

Ackoledgements

I would like to express my gratitude to my supervisor, Miklós Rásonyi, for his guidance and support while writing this thesis. I am also very thankful to him for introducing me to this theory.

I am sincerely thankful to my parents for their constant support and for always encouraging me to follow my dreams.

Köszönetnyilvánítás

Szeretném megköszönni témavezetőmnek, Rásonyi Miklósnak, az érdekes témafelvetést, számos építő hozzászólást és szakmai tanácsot, amellyel hozzájárult a dolgozat elkészítéséhez.

Továbbá szeretném megköszönni a szüleimnek, hogy mindenben támogattak és mindig bátorítottak, hogy kövessem az álmaim.

Contents

Introduction

Merton's optimal investment problem is a well known problem of continuous-time finance, which was named after Nobel laureate Robert Merton's pioneering work in this area. It concerns finding the optimal investment strategy for the investor, who has only two possible objects of investment: a risk-less asset (e. g. a savings account), paying a fixed rate of interest and a number of risky assets (e. g. stock, real estate) whose price is assumed to follow a geometric Brownian motion. Assume that the investor lives for a finite period of time, from present until time T, he starts with an initial amount of money, and he wants to decide how much of which security to hold at each time in order to maximize the final wealth. Naturally, the investor is risk averse to a certain degree, which means that he refrains from investing in assets which have a high risk of loosing money even if it might have high return. The investor's attitude to risk and individual preferences can be characterized by the so called utility functions, this way we do not maximize the expected return of the investment, but the expected utility, making it possible not only to maximize the expected value of the final wealth, but also limit the risk of losing money at the same time.

We consider a continuous-time market model, which means that the investor can re-balance his capital at any moment before the terminal time, that is moving capital from risk-less security to stock and vice versa, without any transaction costs. The assets are supposed to be infinitely divisible, they can be bought or sold in arbitrary amounts anytime. The investor has all the information about the prices at present, but no additional information. Merton formulated and solved the problem (1969, 1971) [3], [4], for the case where there are no transaction costs. Using the methods of stochastic control, he derived a nonlinear partial differential equation (the so called Hamilton-Jacobi-Bellman equation) for the value function of the optimization problem. For some special cases of utility functions (i. e. logarithmic and power utility), closed form of solution was found by him. For these cases, it turns out that the optimal strategy is to keep a constant fraction of the wealth in stocks.

The aim of present thesis to develop better understanding of the above presented problem. It consists of three chapters, first chapter aims to give a brief introduction, we present the financial setting, introduce a method of stochastic control called dynamic programming which we will use later to solve the optimization problem. We also present the solution of the problem for the logarithmic utility function: this case can be treated separately due to the simpler form.

In the second chapter we obtain the theoretical results for power utility: first we derive the optimal strategy to maximize the utility of the consumption throughout the investment and also the final wealth for a finite time horizon, then for an infinite time horizon we maximize the consumption only. The results are indeed theoretical: keeping a constant fraction of the wealth in risky asset (and therefore also a constant fraction in the risk-less one) would mean continuous trading, which is not very realistic. Also, to achieve the optimal result we made the assumption that the asset prices follow a geometric Brownian motion with constant drift and constant volatility. Merton in his original paper already questioned the accuracy of this assumption, however, this model is still the most frequently used financial model.

In the third chapter we would like to check how this strategy works in practice. We take a more general model dropping the assumption of constant drift and check how the Merton strategy works for this case. We would like to answer the following questions: does the expected utility decrease because of discretization? Will an investor who wants to use a Merton-type of strategy, i.e. a constant allocation strategy, get the same constant if we change the model a little bit? Is it better to invest less in the risky asset as uncertainty in the drift of the modified model grows or can an investor profit from it? If we change the model but use the allocation strategy which is optimal in the Black-Scholes case, can we still reach the same expected utility? We will come back to answer these questions in the end of the thesis.

List of Notations and Abbreviations

W_t - Brownian motion

$\mathbf{1}$ - the vector $(1, 1, ..., 1)^\top$

$C^{1,2} = \{f(t,x)|f$ continuously differentiable in t, twice continuously differentiable in $x\}$

$\mathrm{Diag}(V)-$ diagonal matrix with the entries of the vector V in the diagonal

$\mathbb{E}_{t,x}[Y] = \mathbb{E}[Y|X_t = x]$, when it is clear which X we are talking about

$\mathbb{E}_x[Y] = \mathbb{E}_{0,x}[Y]$

$D_x f-$ the gradient of f

$D_{xx} f-$ the Hessian of f

$J(t,x,u)-$ optimization criterion

$\mathcal{A}(t,x)-$ admissible set

$V(t,x)-$ value function

$V(x_0) = V(0, x_0)$

Chapter 1

Background Theory and Formulation of the Problem

This chapter aims to give a brief introduction, we present the financial setting, and we introduce a method of stochastic control called dynamic programming which we will use later to solve the optimization problem. When presenting the basic notions of stochastic finance, we will follow [2] and [8], the part about stochastic control theory is based on [5].

1.1 The Market Model

Let us consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions, (i.e. $\mathcal{F}_0$ contains all the measure 0 sets, and the filtration is right-continuous: $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$, for all t) and let $T > 0$ be a non-random terminal time. $(W_t)_{t\in[0,T]}$ denotes a stochastic process called Brownian motion.

Definition 1. A process $W = (W_t)_{t\geq 0}$ is a $\mathbb{P}$**-Brownian motion** (with respect to $(\mathcal{F}_t)$) if it is $(\mathcal{F}_t)$-adapted and it satisfies

1. W is continuous with $W_0 = 0$ (a.s.),
2. $W_t - W_s$ is independent of $\mathcal{F}_s, 0 \leq s < t$,
3. For any $t > 0$, $W_t \sim N(0,t)$ under the probability measure $\mathbb{P}, 0 \leq s < t$.

A higher dimensional Brownian motion is the vector

$$W_t = (W_t^1, \dots, W_t^n)^\top$$

where the W^i are independent Brownian motions, all adapted to the same filtration $\mathcal{F}$.

As the model for the financial market we will use the **Black-Scholes model**: suppose that the financial market consists of one bond (a riskless bank account paying a fixed interest rate r) with prices

$$dB_t = B_t r dt, B_0 = 1, \text{ that is } B_t = e^{rt},$$

and one stock (risky asset) with prices evolving like

$$dS_t = S_t(\mu dt + \sigma dW_t), S_0 = s_0 > 0,$$

with trend parameter $\mu \in \mathbb{R}$ and volatility $\sigma > 0$. Using Itô's formula, we can derive the explicit solution of the stochastic differential equation:

$$S_t = S_0 \exp((\mu - \frac{\sigma^2}{2})t + \sigma W_t).$$

A such kind of process is said to follow a **geometric Brownian motion**. The wealth (the value of the portfolio) of the investor with initial capital $x_0 > 0$ evolves like

$$dX_t = N_t^B dB_t + N_t^S dS_t, X_0 = x_0,$$

where N_t^B and N_t^S denote the number of the bonds and stocks held by the investor at time t. N_t^B and N_t^S are non-negative, but not necessarily integer.

Suppose that the wealth is always non-negative:

$$X(t) \geq 0 \text{ a.s., for } 0 \leq t \leq T.$$

1.2 Stochastic Control Theory

In this section we will introduce the basic notions of stochastic control theory, following [5].

In this thesis we will consider controls of Itô processes, which satisfy stochastic differential equations driven by Brownian motion.

The SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t,$$

where W denotes the m-dimensional Brownian motion, has a unique strong solution called an Itô diffusion when the drift $b : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and diffusion coefficient $\sigma : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ satisfy the following conditions for all $0 \leq s, t$ and $x, y \in \mathbb{R}^n$

$$||b(s, x) - b(t, y)|| + ||\sigma(s, x) - \sigma(t, y)|| \leq K(||y - x|| + |t - s|)$$
$$||b(t, x)||^2 + ||\sigma(t, x)||^2 \leq K^2(1 + ||x||^2)$$

for some K positive constant.

Definition 2. An $\mathcal{F}$-progressively measurable process $(u_t)_{t\in[0,T]}$ with values in some set $\mathcal{U} \subseteq \mathbb{R}^p$ is called a **control process**. An n-dimensional process $(Y_t)_{t\in[0,T]}$ controlled by u_t if it is defined by

$$dY_t = b(t, Y_t, u_t) + \sigma(t, Y_t, u_t)dW_t, Y_0 = y_0,$$

where $b : [0, T] \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n, \sigma : [0, T] \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^{n\times m}$, $(W_t)_{t\in[0,T]}$ denotes the m-dimensional Brownian motion.

The **optimization criterion** is

$$J(t, x, u) = \mathbb{E}\Big[\int_t^T \psi(t, X_t^u, u_t)dt + \Psi(T, X_T^u)|X_t^u = x\Big]$$

We denote the **admissible set** of controls by $\mathcal{A}(t, x)$ and it contains all the controls, which fulfil the following properties:

1. The control process $u = (u_s)_{s\in[t,T]}$ is progressively measurable with values in $\mathcal{U}$ and $\mathbb{E}[\int_t^T ||u_s||^2 ds] < \infty$.
2. The SDE describing the controlled process has a unique strong solution $(X_s)_{s\in[t,T]}$ with $X_t = x$ and
$$\mathbb{E}_{t,x}[\sup_{t\leq s\leq T} ||X_s||^2] < \infty.$$
3. The optimization criterion $J(t, x, u)$ is well defined.

With the above notations our aim is to maximize the **value function** of the the control problem, that is defined by

$$V(t, x) = \sup_{u\in\mathcal{A}(t,x)} J(t, x, u)$$

and find the optimal value $V(0, x_0)$ as well as the optimal control strategy u^* for which this value is attained, that is $V(0, x_0) = J(0, x_0, u^*)$.

1.2.1 Dynamic programming

The idea of dynamic programming is to break the optimization problem down to smaller sub-problems and use these result to get the overall optimum. In this section we briefly describe how this method works and later we are going to use this to solve the portfolio optimization problem. In order to be able to use dynamic programming, the problem needs to have a specific optimal substructure. This property is called the Bellman Principle:

$$V(t, x) = \sup_{u\in\mathcal{A}(t,x)} \mathbb{E}_{t,x}\Big[\int_t^{t_1} \psi(s, X_s^u, u_s)ds + V(t_1, X_{t_1}^u)\Big]$$

This principle basically states that we can isolate a part of the optimization problem, an optimal control on an interval $[t, t_1]$ will stay optimal if at t_1 we continue optimally.

One way to solve an optimal control problem is to use the Bellman principle (it needs to be proved that it holds). If the wealth process has sufficient smoothness properties, we can apply Itô formula to $V(t_1, X^u_{t_1})$ and put it back to the above equation, yielding

$$\begin{aligned} V(t,x) = \sup_{u \in \mathcal{A}(t,x)} \mathbb{E}_{t,x}\Big[\int_t^{t_1} & \psi(s, X_s, u_s) ds + V(t, X_t) \\ &+ \int_t^{t_1} V_t(s, X_s)(D_x V(s, X_s))^\top b(s, X_s, u_s) ds \\ &+ \int_t^{t_1} (D_x V(s, X_s))^\top \sigma(s, X_s, u_s) dWs \\ &+ \frac{1}{2} \int_t^{t_1} tr((D_{xx} V(s, V_s))^\top \sigma(s, X_s, u_s) \sigma(s, X_s, u_s)^\top \Big] \end{aligned}$$

The stochastic integral part $\int_t^{t_1} (D_x V(s, X_s))^\top \sigma(s, X_s, u_s) dWs$ is a martingale, so its expectation is 0. Using the notation

$$a(s, X_s, u_s) = \sigma(s, X_s, u_s)\sigma(s, X_s, u_s)^\top$$

for the diffusion matrix we obtain

$$\begin{aligned} V(t,x) = \sup_{u \in \mathcal{A}(t,x)} \mathbb{E}_{t,x}\Big[\int_t^{t_1} & \psi(s, X_s, u_s) ds + V(t, X_t) \\ &+ \int_t^{t_1} V_t(s, X_s)(D_x V(s, X_s))^\top b(s, X_s, u_s) ds \\ &+ \frac{1}{2} \int_t^{t_1} tr((D_{xx} V(s, V_s))^\top a(s, X_s, u_s) \Big]. \end{aligned}$$

Let us subtract $V(t, x)$ from the equation and divide by $(t_1 - t)$ and let t_1 tend to t. Now we have to check whether taking limit and expectation can be interchanged, as well as taking supremum and limit. If so, since we take conditional expectation when $X_t = x$, it follows that $V(t, X_t) = V(t, x)$. Then we get that

$$\begin{aligned} 0 = \sup_{u \in \mathcal{U}} \Big\{ \Psi(t, x, u) + V_t(t, x) + (D_x V(t, x))^\top b(t, x, u) \\ + \frac{1}{2} tr((D_{xx} V(t, x)) a(t, x, u)) \Big\} \end{aligned}$$

We define a differential operator (which depends on u) for $\mathcal{C}^{1,2}$ functions

$$\mathcal{L}^u f(t, x) = V_t(t, x) + (D_x f(t, x))^\top b(t, x, u) + \frac{1}{2} tr((D_{xx} f(t, x)) a(t, x, u))$$

and we can rewrite the above equation as

$$0 = \sup_{u \in \mathcal{U}} \{\Psi(t, x, u) + \mathcal{L}^u V(t, x)\}$$

This equation is called **Hamilton-Jacobi-Bellman** equation (or HJB equation). Now we have derived a necessary condition, since we have seen that the value function V solves the Hamilton-Jacobi-Bellman equation under certain conditions.

Now we want to formulate a condition which ensures that the solution found is indeed the value function and it provides an optimal control strategy.

Theorem 3. (Verification Theorem) Let σ satisfy the growth condition $||\sigma(t, x, u)||^2 \leq C_\sigma(1 + ||x||^2 + ||u||^2)$, and let ψ be continuous with $||\psi(t, x, u)||^2 \leq C_\psi(1 + ||x||^2 + ||u||^2)$ for C_σ, C_ψ positive constants for all $t \in \mathbb{R}_+, x \in \mathbb{R}^n, u \in \mathcal{U}$.

1. Let $\Phi \in \mathcal{C}^{1,2}([0, T) \times \mathbb{R}^n)$ be continuous on $[0, T] \times \mathbb{R}$ with $||\Phi(t, x)|| \leq C_\Phi(1 + ||x||^2)$ for a positive constant C_Φ and

$$\sup_{u \in \mathcal{U}} \{\Phi(t, x, u) + \mathcal{L}^u \Phi(t, x)\} = 0, \text{ for } t \in [0, T), x \in \mathbb{R}^n$$
$$\Phi(T, x) = \Psi(t, x), x \in \mathbb{R}^n$$

   Then for every $t \in [0, T], x \in \mathbb{R}^n$

$$\Phi(t, x) \geq V(t, x).$$

2. If there exists a maximizer $\hat{u}(t, x)$ of the function $u \mapsto \psi(t, x, u) + \mathcal{L}^u \Phi(t, x)$ such that $u_t^* = \hat{u}(t, X_t^*)$ is admissible, $u^* = (u_t^*)_{t \in [0,T]}$, then $\Phi(t, x) = V(t, x)$ for $t \in [0, T), x \in \mathbb{R}^n$ and the optimal control strategy is given by u^*.

This means that $V(t, x) = J(t, x, u^{t,x})$, where $u^{t,x} = (u_s^*)_{s \in [t,T]}$ and X_t^* denotes the solution of the SDE describing the controlled process, for control u_s^*.

Proof. Let us fix $t \in [0, T], x \in \mathbb{R}^n$ and define the following hitting times

$$\tau_n = \min\{T, \inf\{s > t : ||X_s - X_t|| \geq n\}\},$$

this way we can argue for bounded processes. Using Itô formula for X_t we get

$$\Phi(\tau_n, X_{\tau_n}) = \Phi(t, x) + \int_t^{\tau_n} \mathcal{L}^{u_s} \Phi(s, X_s) ds + \int_t^{\tau_n} \Phi_x(s, X_s)^\top \sigma(s, X_s, u_s) dW_s$$

Finiteness of $\mathbb{E}_{t,x}[\int_t^{\tau_n} ||\Phi_x(s,X_s)^\top \sigma(s,X_s,u_s)||^2 ds$ follows from the continuity of Φ and boundedness of X, therefore

$$\mathbb{E}_{t,x}\Big[\int_t^{\tau_n} \Phi_x(s,X_s)^\top \sigma(s,X_s,u_s)dW_s\Big] = 0.$$

Then

$$\begin{aligned}
&\mathbb{E}_{t,x}\Big[\int_t^{\tau_n} \psi(s,X_s,u_s)ds + \Phi(\tau_n, X_{\tau_n})\Big] \\
&\quad = \mathbb{E}_{t,x}\Big[\int_t^{\tau_n} \psi(s,X_s,u_s)ds + \Phi(t,x) + \int_t^{\tau_n} \mathcal{L}^{u_s}\Phi(s,X_s)ds\Big] \\
&\quad = \Phi(t,x) + \mathbb{E}_{t,x}\Big[\int_t^{\tau_n} (\psi(s,X_s,u_s) + \mathcal{L}^{u_s}\Phi(s,X_s))ds\Big]
\end{aligned}$$

Ψ satisfies the HJB, i.e.

$$\sup_{u\in\mathcal{U}}\{\Phi(s,x,u) + \mathcal{L}^u\Phi(s,x)\} = 0, \text{ for } t \in [0,T), x \in \mathbb{R}^n,$$

therefore

$$(\psi(s,X_s,u_s) + \mathcal{L}^{u_s}\Phi(s,X_s)) \leq 0 \text{ for all } s \in [t,T],$$

yielding

$$\mathbb{E}_{t,x}\Big[\int_t^{\tau_n} \psi(s,X_s,u_s)ds + \Phi(\tau_n, X_{\tau_n})\Big] \leq \Phi(t,x).$$

As n tends to infinity $\tau_n \to T$. Then, using dominated convergence, we get

$$\mathbb{E}_{t,x}\Big[\int_t^{\tau_n} \psi(s,X_s,u_s)ds + \Phi(\tau_n, X_{\tau_n})\Big] \to J(t,x,u), \text{ as } n \to \infty,$$

as an integrable dominating function we can use the growth conditions on Φ and ψ to define

$$\Big|\int_t^{\tau_n} \psi(s,X_s,u_s)ds + \Phi(\tau_n, X_{\tau_n})\Big| \leq C_\psi \int_t^T (1+||X_s||^2+||u_s||^2)ds + C_\Phi(1+||X_T||^2) \in L^1.$$

Then we get that also

$$J(t,x,u) \leq \Phi(t,x),$$

and taking supremum in u

$$V(t,x) \leq \Phi(t,x).$$

To prove part (2.), let us observe that if we manage to find an optimizer $\hat{u}(t,x)$ and use the strategy $u_t^* = (t, X_t^*)$, then we get equality instead of inequality, so using similar arguments we get

$$V(t,x) = \Phi(t,x)$$

□

To sum up, to solve the optimization problem we will proceed as follows:

1. Find an optimal $\hat{u}$ for the HJB equation.

2. If we find a such $\hat{u}$, it will depend on $V_t, D_xV, D_{xx}V$ formally, so it will have the form
$$\hat{u}(t,x) = \hat{u}(t,x,V_t,D_xV,D_{xx}V).$$
Then we plug in the value we got for u in the HJB equation to get a partial differential equation for V. Solving this equation for the boundary condition $V(T,x) = \Psi(T,x)$ we get a V^* which is our candidate for the value function.

3. Use the verification theorem to check that the function V we got from the HJB and the maximizer $\hat{u}$ are indeed the value function optimal and the optimal control strategy.

1.3 Utility Functions

In order to characterize the investor's decisions and preferences we need the concept of utility functions. Utility function basically expresses how satisfied the investor is with a certain outcome of the investment, this way it is a function of the wealth or a function of the consumption.

We assume the investor to be risk averse, meaning that he will only accept investments which are "better than fair game", this implies strict concavity of the utility function. The second assumption we make is that he will always prefer more wealth to less, this can be referred to as non-satiation of the investor and it implies that the utility function is strictly monotone increasing.

Definition 4. For a subset $S \subseteq \mathbb{R}$, $U : S \to \mathbb{R}$ is a **utility function**, if U is strictly increasing, strictly concave and continuous on S.

Definition 5. For a utility function $U : S \to \mathbb{R}$ the **absolute risk aversion coefficient** is defined to be the ratio
$$A(x) = -\frac{U''(x)}{U'(x)},$$
respectively the **relative risk aversion coefficient** is the ratio
$$R(x) = -\frac{xU''(x)}{U'(x)}$$

In this thesis we will use two different utility functions, logarithmic utility $U(x) = \log(x)$, log denoting the natural logarithm, and power utility $U(x) = \frac{x^\gamma}{\gamma}$, where $\gamma \in \mathbb{R}, \gamma < 1$, $\gamma \neq 0$. In fact, the logarithmic case corresponds to $\gamma = 0$. These two classes of functions are the so called hyperbolic absolute risk aversion (HARA) or also referred to as constant relative risk aversion (CRRA) class.

Coming back to the portfolio optimization problem,a s control at time t we will use the fraction u_t of the total wealth which should be invested in risky assets. Then

$$N_t^B = \frac{(1-u_t)X_t}{B_t}, N_t^S = \frac{u_t X_t}{S_t}$$

then we can rewrite the wealth process in the following way

$$\begin{aligned} dX_t &= (1-u_t)X_t r dt + u_t X_t(\mu dt + \sigma dW_t) \\ &= X_t((r + u_t(\mu - r))dt + u_t \sigma dW_t). \end{aligned}$$

1.4 First Example

In the case of logarithmic utility function we can derive the solution in a simpler way for the case when there is no consumption, meaning that we only want to maximize the expected utility of the final wealth, without consuming money from the bank account throughout the investment period.

Our aim is to solve the following optimization problem:

$$\max_u \mathbb{E}[U(X_T)|X_0 = x_0]$$

Theorem 6. For $U(x) = \log x$ the optimal policy is

$$u_t^* = \frac{\mu - r}{\sigma^2}, \text{ for all } t \in [0, T].$$

Proof. We need to solve the following SDE:

$$dX_t = X_t((r + u_t(\mu - r))dt + u_t \sigma dW_t).$$

To do so, guessing

$$X_t = x_0 \exp\{\int_0^t g_s ds + \int_0^t h_s dW_s\},$$

applying Itô-formula we get

$$\begin{aligned} dX_t &= X_t g_t dt + X_t h_t dW_t + \frac{1}{2} X_t g_t^2 dt \\ &= X_t((g_t + \frac{1}{2}h_t^2)dt + h_t dW_t) \end{aligned}$$

then comparing the coefficients we get $g_t + \frac{1}{2}h_t^2 = r + (\mu - r)u_t$, $h_t = \sigma u_t$. Thus we get

$$X_t = x_0 \exp\{\int_0^t (r + (\mu - r)u_s - \frac{1}{2}\sigma^2 u_s^2)ds + \int_0^t \sigma u_s dW_s\}$$

Our aim is to maximize

$$\mathbb{E}[\log(X_T^u)|X_0 = x_0]$$

over all admissible control strategies in

$$\mathcal{A}(x_0) = \{u : \mathbb{E}\int_0^t (|bu_t| + |\sigma u_t|^2)dt < \infty, X_t^u > 0, \mathbb{E}[(\log X_T)^-] < \infty\}.$$

We have that $\int \sigma u_t dW_t$ is a martingale, thus in particular

$$\mathbb{E}\Big[\int_0^T \sigma u_t dW_t\Big] = 0.$$

So for every $u \in \mathcal{A}(x_0)$

$$J(x_0, u) = \log x_0 + \mathbb{E}\Big[\int_0^T (r + (\mu - r)u_s - \frac{1}{2}\sigma^2 u_s^2)ds\Big].$$

We can check that $(r + (\mu - r)u_s - \frac{1}{2}\sigma^2 u_s^2)$ is strictly concave, by taking derivatives

$$\frac{\partial}{\partial u}((r + (\mu - r)u_s - \frac{1}{2}\sigma^2 u_s^2)) = \mu - r - \sigma^2 u$$
$$\frac{\partial^2}{\partial u^2}((r + (\mu - r)u_s - \frac{1}{2}\sigma^2 u_s^2)) = -\sigma^2 < 0.$$

Setting $\mu - r - \sigma^2 u = 0$ we get that the unique optimal value for u_t is

$$u_t^* = \frac{\mu - r}{\sigma^2},$$

with the corresponding value function

$$V(0, x_0) = J(0, x_0, u^*) = log(x_0) + (r + \frac{\mu - r}{2\sigma^2})T$$

□

This means that the optimal investment strategy is to keep the same, fixed proportion of the total wealth invested in stocks, e.g. if the optimal policy $u_t = 0.2$ then the investor should always keep 20% of the wealth is the risky asset. This strategy is not doable in practise: it would mean constant trading. In Chapter 3. we will see how close we can get to the optimal expected utility by discrete trading.

The simplicity of this solution is due to the fact that the SDE had an exponential solution, thus taking the logarithm leads to a simple equation which we were able to maximize pointwise. This method would not work for another type of utility function.

The following step would be the case, where there is not only one stock to choose. We basically get the same result for this case, as stated below, the optimal strategy is keeping a constant fraction of the wealth in stocks.

Let us consider n stocks with prices $(S_t)_{t\in[0,T]}$, $S_t = (S_t^1, ..., S_t^n)^\top$, with dynamics

$$dS_t = Diag(S_t)(\mu dt + \sigma dW_t), S_0 = s_0,$$

where $s_0^i > 0$ for $I = 1, 2, ..., n$, $\mu \in \mathbb{R}^n$, and σ a non-singular volatility matrix in $\mathbb{R}^{n\times n}$. So stock i evolves like

$$dS_t^i = S_t^i \left(\mu_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_t^j \right), i = 1, ..., n.$$

As control at time t, we will take an n-dimensional process u, where u_t^i denotes the fraction of wealth which is invested in stock i.

The optimal solution is

$$u_t^* = \pi^* = (\sigma\sigma^\top)^{-1}(\mu - r), t \in [0, T].$$

Chapter 2

The Main Result

Let us now assume that the investor does not only want to maximize the expected utility of the final wealth, but also wants to make a living from the investment and he consumes money at rate $c(t)$ from the bank account. This way, his objective is to maximize the expected utility of the consumption throughout the investment period. In this chapter we will follow [5].

2.1 Utility of Consumption

We consider a similar financial market consisting of one bond with prices evolving like

$$dB_t = B_t r dt, B_0 = 1$$

$$\text{that is } B_t = e^{rt},$$

and n stocks with prices

$$dS_t = Diag(S_t)(\mu dt + \sigma dW_t), S_0 = s_0,$$

where W is an m-dimensional Brownian motion, $m \geq n, s_0^i > 0$ for $i = 1, ..., n, \mu \in \mathbb{R}^n$, and $\sigma \in \mathbb{R}^{n \times m}$ with maximal rank, implying that the matrix $\sigma\sigma^\top \in R^{n \times n}$ is not singular.So stock i evolves like

$$dS_t^i = S_t^i \left(\mu_i dt + \sum_{j=1}^{m} \sigma_{ij} dW_t^j \right), i = 1, ..., n.$$

As controls $u = (\pi, c)$ we take the vector of fractions $\pi_t = (\pi_t^1, ..., \pi_t^n)^\top$, where π_t^i denotes fraction of the wealth invested in stock i, and c_t stands for the consumption rate. Then the wealth process is evolving like

$$dX_t^u = \sum_{i=1}^n X_t^u \pi_t^i dS_t^i / S_t^i + X_t^u (1-\pi_t) dB_t / B_t - c_t dt$$
$$= ((r + \pi_t^\top (\mu - r\mathbf{1})) X_t^u - c_t) dt + X_t^u \pi_t^\top \sigma s W_t,$$

The investor assigns utility $U_1(c_t)$ to the consumption rate c_t and utility $U_2(X_T^u)$ to the wealth at terminal time T. U_1 and U_2 could be different, but we will consider power utility function for both, i.e.

$$U(x) = \frac{x^\alpha}{\alpha}, \alpha < 1, \alpha \neq 0$$

and a discounting factor $e^{-\beta t}, \beta \geq 0$. Thus we want to maximize

$$J(t,x,u) = \mathbb{E}_{t,s}\Big[\int_t^T e^{-\beta t} U(c_t) dt + e^{-\beta T} U(X_T^u)\Big].$$

Theorem 7. For $U(x) = \frac{x^\alpha}{\alpha}$ and the above presented optimization problem, the optimal policy pair (c_t^*, π_t^*) is

$$c_t^* = e^{-\frac{\beta t}{1-\alpha}} h(t)^{-1} X_t^*$$
$$\pi_t^* = \frac{1}{1-\alpha} (\sigma\sigma^\top)^{-1} (\mu - r\mathbf{1})$$

Proof. We want to proceed using dynamic programming.

The corresponding partial differential operator is

$$\mathcal{L}^{(\pi,c)} v(t,x) = v_t(t,x) + ((r + \pi^\top(\mu - r\mathbf{1}))x - c) v_x(t,x) + \frac{1}{2} \pi^\top \sigma \sigma^\top \pi x^2 v_{xx}(t,x)$$

with the Hamilton-Jacobi-Bellman equation

$$\sup_{\pi,c} \Big\{ e^{-\beta t} \frac{c^\alpha}{\alpha} + \mathcal{L}^{(\pi,c)} V(t,x) \Big\} = 0.$$

When V is increasing and concave and x positive, we get the following maximizers

$$c(t,x) = (e^{\beta t} V_x(t,x))^{\frac{1}{\alpha-1}}$$
$$\pi(t,x) = -(\sigma\sigma^{top})^{-1} (\mu - r\mathbf{1}) \frac{V_x(t,x)}{x V_{xx}}$$

Putting this back to the Hamilton-Jacobi-Bellman equation, we have to solve

$$\frac{1-\alpha}{\alpha} e^{-\frac{\beta t}{1-\alpha}} V_x^{\frac{\alpha}{\alpha-1}} + V_t + r x V_x - \frac{1}{2} (\mu - r\mathbf{1})^\top (\sigma\sigma^\top)^{-1} (\mu - r\mathbf{1}) \frac{V_x^2}{V_{xx}} = 0$$

at terminal time we have boundary condition $V(T,x) = e^{-\beta T} \frac{x^\alpha}{\alpha}$.

Guessing

$$V(t,x) = h(t)^{1-\alpha} \frac{x^\alpha}{\alpha}$$

we get $h(T) = e^{-\frac{\beta T}{1-\alpha}}$ boundary condition at terminal time for h and

$$e^{-\frac{\beta t}{1-\alpha}} + ch(t) + h'(t) = 0,$$

where

$$c = \frac{\alpha}{1-\alpha}\Big(r + \frac{1}{2(1-\alpha)}(\mu - r\mathbf{1})^\top(\sigma\sigma^\top)^{-1}(\mu - r\mathbf{1})\Big).$$

This is a linear ODE and for $\beta - (1-\alpha)c \neq 0$ it has the solution

$$h(t) = e^{-\frac{\beta t}{1-\alpha}}e^{-c(T-t)} + \frac{(1-\alpha)e^{-ct}}{\beta - (1-\alpha)c}\Big\{e^{-\frac{\beta-(1-\alpha)c}{1-\alpha}t} - e^{-\frac{\beta-(1-\alpha)c}{1-\alpha}T}\Big\},$$

for $\beta - (1-\alpha)c = 0$

$$h(t) = e^{-ct}(1 + T - t).$$

It is easy to see that this solution for h takes strictly positive values for every t. Then, since $V(t,x) = h(t)^{1-\alpha}\frac{x^\alpha}{\alpha}$, if the equation for the controlled process (for controls c, π defined above) has a positive unique solution, then the value function V is also positive. This means that $V \in \mathcal{C}^{1,2}$, furthermore, monotonicity and concavity of V follow from the sign of the derivatives

$$V_x(t,x) = h(t)^{1-\alpha}x^{\alpha-1} > 0$$
$$V_{xx}(t,x) = -(1-\alpha)h(t)^{1-\alpha}x^{\alpha-2} < 0$$

Substituting $V(t,x) = h(t)^{1-\alpha}\frac{x^\alpha}{\alpha}$ into the equations for c and π we get

$$c(t,x) = e^{-\frac{\beta t}{1-\alpha}}h(t)^{-1}x$$
$$\pi(t,x) = \frac{1}{1-\alpha}(\sigma\sigma^\top)^{-1}(\mu - r\mathbf{1})$$

For the control defined by the policy pair (c_t^*, π_t^*), where

$$c_t^* = c(t,x) = e^{-\frac{\beta t}{1-\alpha}}h(t)^{-1}X_t^*$$
$$\pi_t^* = \pi(t,x) = \frac{1}{1-\alpha}(\sigma\sigma^\top)^{-1}(\mu - r\mathbf{1})$$

we get that the controlled process X_t^* is of form

$$\begin{aligned} dX_t^* &= ((r + \pi_t^\top(\mu - r\mathbf{1}))X_t^u - c_t)dt + X_t^*\pi_t^\top\sigma sW_t \\ &= X_t^*((r + \pi_t^\top(\mu - r\mathbf{1}) - e^{-\frac{\beta t}{1-\alpha}}h(t)^{-1})dt + \pi_t^\top\sigma sW_t), \end{aligned}$$

so the unique strong solution for this SDE is a stochastic exponential, hence strictly positive. To use the verification theorem we need integrability conditions, which are satisfied because the solution is strong. Admissibility and growth conditions can be checked using a a suitable set $\mathcal{U}$ of controls, which, being partly difficult, is out of the scope of this thesis.

This means that the value function is indeed how we defined it and (c_t^*, π_t^*) is indeed the optimal control strategy. □

2.2 Discounted Utility of Consumption

We now want to maximize the the discounted utility of consumption in the same model, that is

$$J(x,u) = E_x\Big[\int_0^\infty e^{-\beta t}\frac{c_t^\alpha}{\alpha}dt\Big]$$

over all admissible policy pairs $u = (\pi, c)$ that control the process

$$dX_t = ((r + \pi_t^\top(\mu - r\mathbf{1}))X_t - c_t)dt + X_t\pi_t^\top \sigma dW_t,$$

$$X_0 = x.$$

Note that in this case we take an infinite period of time: final wealth has no utility, we only care about the utility of consumption.

Theorem 8. For $U(x) = \frac{x^\alpha}{\alpha}$ and the above presented optimization problem, the optimal policy pair (c_t^*, π_t^*) is

$$\pi_t^* = \frac{1}{1-\alpha}(\sigma\sigma^\top)^{-1}(\mu - r\mathbf{1})$$
$$c_t^* = A^{\frac{1}{\alpha-1}}X_t^*,$$

where $A = \left(\frac{\alpha}{1-\alpha}\left(\frac{\beta}{\alpha} - r - \frac{1}{2(1-\alpha)}(\sigma\sigma^\top)^{-1}(\mu - r\mathbf{1})^\top\sigma\sigma^\top(\sigma\sigma^\top)^{-1}(\mu - r\mathbf{1})\right)\right)^{\alpha-1}$

Proof. Let $\beta > 0$, $\alpha \in (0,1)$, initial wealth $x > 0$, and we also assume that $\mathbb{P}(X_t > 0) = 1$ for every t>0. The value function is

$$V(x) = \sup_{u\in\mathcal{A}(x)} J(x,u),$$

and the corresponding Hamilton-Jacobi-Bellman equation reads as

$$\sup_{u\in\mathbb{R}^n\times[0,\infty)}\Big\{((r + \pi^\top(\mu - r\mathbf{1}))x - c)V_x + \frac{1}{2}\pi^\top\sigma\sigma^\top\pi x^2 V_{xx} - \beta V + \frac{c^\alpha}{\alpha}\Big\} = 0.$$

Supposing $V_x > 0$ and $V_{xx} < 0$, we get the following optimum:

$$\pi = -\nu\frac{V_x(x)}{xV_{xx}(x)}, \text{ where } \nu = (\sigma\sigma^\top)^{-1}(\mu - r\mathbf{1}),$$

and

$$c(x) = V_x(x)^{\frac{1}{\alpha-1}}.$$

Putting these values in the HJB equitation, we get the ODE

$$-\frac{1}{2}\nu^\top\sigma\sigma^\top\nu\frac{V_x^2}{V_{xx}} + rxV_x - \beta V + \frac{1-\alpha}{\alpha}V_x^{\frac{\alpha}{\alpha-1}} = 0.$$

Guessing $V(x) = A\frac{x^\alpha}{\alpha}$ for some $A > 0$ yields $V_x(x) = Ax^{\alpha-1}$, $V_{xx}(x) = -(1-\alpha)Ax^{\alpha-2}$, so we need to solve

$$\frac{1}{2(1-\alpha)}\nu^\top\sigma\sigma^\top\nu + r - \frac{\beta}{\alpha} + \frac{1-\alpha}{\alpha}A^{\frac{1}{\alpha-1}} = 0.$$

for $x > 0$. So we need to assume that

$$\beta > \frac{\alpha}{2(1-\alpha)}\nu^\top\sigma\sigma^\top\nu + \alpha r,$$

because A was assumed to be strictly positive. Solving the equation for A we get

$$A = \Big(\frac{\alpha}{1-\alpha}\Big(\frac{\beta}{\alpha} - r - \frac{1}{2(1-\alpha)}\nu^\top\sigma\sigma^\top\nu\Big)\Big)^{\alpha-1}.$$

So we get the following candidates for the optimal policy

$$\pi_t^* = \frac{1}{1-\alpha}(\sigma\sigma^\top)^{-1}(\mu - r\mathbf{1})$$
$$c_t^* = A^{\frac{1}{\alpha-1}}X_t^*$$

and the wealth process, controlled by this policy pair, is evolving like

$$dX_t^* = \frac{1}{1-\alpha}X_t^*((1-\alpha)r + \nu^\top(\sigma\sigma^\top\nu - r\mathbf{1}) - (1-\alpha)A^{\frac{1}{\alpha-1}})dt + \nu^\top\sigma dW_t).$$

Solving the SDE gives us the unique strong solution

$$X_t^* = X_0\exp\Big\{\Big(\Big(1-\frac{1}{1-\alpha}\nu^\top\mathbf{1}\Big)r + \frac{1-2\alpha}{2(1-\alpha)^2}\nu^\top\sigma\sigma^\top\nu - A^{\frac{1}{\alpha-1}}\Big)t + \frac{1}{1-\alpha}\nu^\top\sigma W_t\Big\}.$$

Since $X_0 > 0$, the solution is also strictly positive, then it follows that $V_x > 0$ and $V_{xx} < 0$. We want to use the verification theorem to argue that we indeed have found the optimum. V is clearly twice continuously differentiable. The integrability conditions follow from the fact that (X_t) is an L^2 process. □

Chapter 3

Simulations

As we have already seen in the previous chapter, the best strategy for the investor is to keep a constant fraction of the wealth in the risky asset. To achieve this result we made the assumption that the asset prices follow a geometric Brownian motion. Merton in his original paper already questioned the accuracy of the above assumption, however, this model is still the most frequently used financial model. In this chapter we would like to study how the Merton-strategy works if we consider a more general market model.

3.1 Fads Models

The Fads-models were introduced by Summers [9] and Shiller [6] in the eighties. These models can be considered as modified Black-Scholes models where we do not assume that the drift is constant. For a positive constant ε let Y_ε be an Ornstein-Uhlenbeck process defined by the following SDE:

$$dY_\varepsilon(t) = -\frac{1}{\varepsilon} Y_\varepsilon(t)dt + dW(t)$$

$$Y_\varepsilon(0) = 0$$

Ornstein-Uhlenbeck processes, and this specific one in particular, have mean reverting property: the drift part of the process depends on the current value, and as the white-noise term given by $dW(t)$ draws the process around, the drift pulls it back to the mean, 0 in this case. Plots below show that the smaller ε is, the more the process is pulled back to 0. Above stochastic differential equation has the explicit solution

$$Y_\varepsilon(t) = \int_0^t e^{-(t-s)\frac{1}{\varepsilon}} dW(s).$$

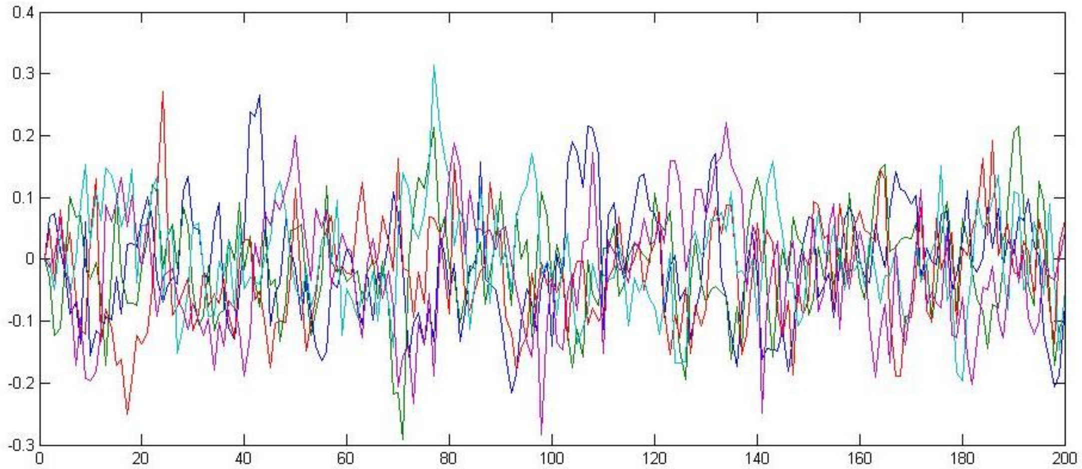

Figure 3.1: 5 plots of Ornstein-Uhlenbeck process for $\varepsilon = 0.01$

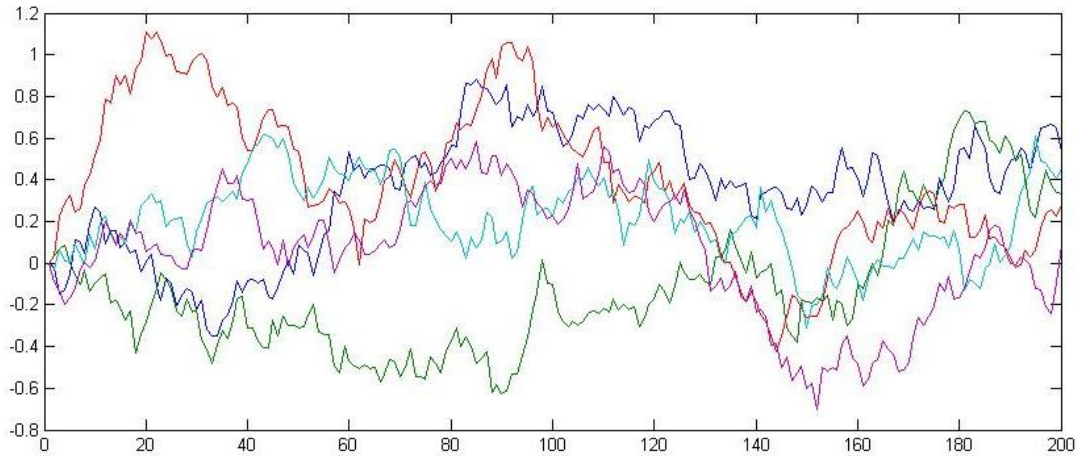

Figure 3.2: 5 plots of Ornstein-Uhlenbeck process for $\varepsilon = 1$

We can now introduce a modified Black-Scholes model as follows:

$$dS(t) = S(t)((\mu + Y_\varepsilon(t))dt + \sigma\rho dW(t) + \sigma\sqrt{1-\rho^2}dB(t)),$$

where $B(t)$ and $W(t)$ are two independent Brownian motions and $0 < \rho < 1$. For $\varepsilon = 0$ this is exactly the original Black-Scholes model, but but $\varepsilon \neq 0$ it clearly does not have constant drift.

3.2 The Simulation Model

We will use the method of discretization to simulate the solution trajectories of the stochastic differential equations. These trajectories will be the asset prices. Let T be fixed and let us take N discretization steps,

$0 = t_0 < t_1 < ... < t_N = T$, $dt = T/N$. We know the analytic solution of SDEs describing both the geometric Brownian motion X_t and Ornstein-Uhlenbeck Y_t, so these processes can be modeled as follows:

$$X_t = X_0 \exp((\mu - \frac{\sigma^2}{2})t + \sigma W_1(t))$$

yielding the discretized version:

$$X(t_n) = X_0(\exp(\sum_{j=0}^{n-1}(\mu - \frac{1}{2}\sigma^2)dt + \sigma\sqrt{dt}Z_1(j))).$$

$$Y_{\varepsilon,t} = \int_0^t e^{-(t-s)\frac{1}{\varepsilon}} dB(s),$$

$$Y_\varepsilon(t_n) = e^{\frac{-t_n}{\varepsilon}} \sum_{j=1}^{n-1} e^{\frac{t_j}{\varepsilon}} \sqrt{dt} Z_2(j)$$

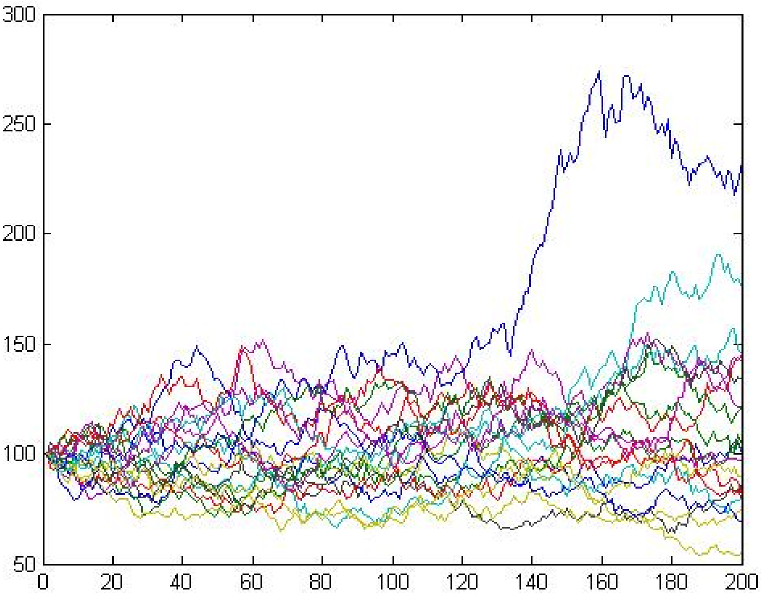

Figure 3.3: 20 plots of geometric Brownian motion for parameters $\mu = 0.12, \sigma = 0.4, X_0 = 100$ with $T = 1$ and 200 discretization steps

For the solution trajectories of the Fads model we have:

$$dS(t) = S(t)((\mu + Y_\varepsilon(t))dt + \sigma\rho dW(t) + \sigma\sqrt{1-\rho^2}dB(t)),$$

which can be simulated by discretization as follows:

$$S(t_{n+1}) = S(t_n)(\mu + Y_\varepsilon(t))dt + \sigma\rho\sqrt{dt}Z_3(n) + \sigma\sqrt{1-\rho^2}\sqrt{dt}Z_2(n),$$

where W_1, W an B are independent Brownian motions, thus Z_1, Z_2, Z_3 are vectors of independent $N(0,1)$ random variables. Once the asset prices are simulated, we can get the wealth process for a trading strategy.

3.3 Finding the Best Constant Allocation Strategy

In this section we study which is the best constant allocation strategy for the Fads-model for different values of ε. The ratio of the wealth invested in stock is a value between 0 and 1, so we take a mesh of the $[0,1]$ interval and compute the expected utility of final wealth using this constant trading strategy.

This case resembles an investor, who is aware of the fact that the market is not a Black-Scholes market, still he wants to use a Merton-type of strategy, i.e. a constant allocation strategy.

3.3.1 Logarithmic utility

The following table contains the simulation results for logarithmic utility, initial capital $V_0 = 500$, initial stock price $S_0 = 100, \rho = 0.3$, MR and OV being the theoretically obtained Merton-ratio and optimal value, Sim. MR and Sim. OV respectively the values obtained by simulation. For these results we took $N = 1000$ discretization steps (this means also re-balancing the wealth 1000 times) and $M = 1000000$ trajectories of each process.

Model	μ	σ	r	ε	MR	Sim. MR	OV	Sim OV
BS	0.12	0.4	0.07	-	0.3125	0.312	6.4409	6.2923
Fads	0.12	0.4	0.07	0.01	-	0.308	-	6.2921
Fads	0.12	0.4	0.07	0.1	-	0.307	-	6.2921
Fads	0.12	0.4	0.07	1	-	0.305	-	6.2920
BS	0.5	1	0.1	-	0.4	0.4	6.5146	6.3952
Fads	0.5	1	0.1	0.01	-	0.4	-	6.3944
Fads	0.5	1	0.1	0.1	-	0.399	-	6.3939
Fads	0.5	1	0.1	1	-	0.397	-	6.3935

From the first row we can see that that in the case of Black-Scholes model, we get back the Merton-ratio we obtained theoretically, but the simulated expected utility of the final wealth will always be a lower estimation of the true optimal value, since continuous trading is not possible.

Now looking at the constants we got for the best allocation strategies, we can see that it slightly decreases as ε grows (we can also think of the Black-Scholes case as the Fads-model with $\varepsilon = 0$). The interpretation for this phenomenon would be that, although an investor with logarithmic utility function is not very risk-averse, as the uncertainty in the drift grows, he will invest less and less in the risky asset in order to control the downside risk.

3.3.2 Power utility

The following table contains the simulation results for power utility function, with exponent γ, initial capital $V_0 = 500$, initial stock price $S_0 = 100, \rho = 0.3$, MR and OV being the theoretically obtained Merton-ratio and optimal value, Sim. MR and Sim. OV respectively the values obtained by simulation. For these results we took $N = 1000$ discretization steps (this means also re-balancing the wealth 1000 times) and $M = 1000000$ trajectories of each process.

Model	μ	σ	r	γ	ε	MR	Sim. MR	OV	Sim OV
BS	0.12	0.4	0.07	0.1	-	0.3472	0.347	18.7098	18.7635
Fads	0.12	0.4	0.07	0.1	0.01	-	0.347	-	18.7632
Fads	0.12	0.4	0.07	0.1	0.1	-	0.411	-	18.7637
Fads	0.12	0.4	0.07	0.1	1	-	0.411	-	18.7658
BS	0.12	0.4	0.07	0.5	-	0.6250	0.625	46.7798	46.6767
Fads	0.12	0.4	0.07	0.5	0.01	-	0.632	-	47.4220
Fads	0.12	0.4	0.07	0.5	0.1	-	1	-	48.0235
Fads	0.12	0.4	0.07	0.5	1	-	1	-	48.0235
BS	0.5	1	0.1	0.1	-	0.444	0.444	18.8768	18.9706
Fads	0.5	1	0.1	0.1	0.01	-	0.444	-	18.9696
Fads	0.5	1	0.1	0.1	0.1	-	0.461	-	18.9702
Fads	0.5	1	0.1	0.1	1	-	0.461	-	18.9759

We can see that in all of the above cases it is better for the investor to invest more in the risky asset as uncertainty in the drift grows. Not only the Merton-ratio grows, but we can see a significant growth in the expected utility as well.

Note that these investors are even less risk averse than the previous one, e.g. while the investor using logarithmic utility function invest 30% of the wealth in the risky asset for the parameters $\mu = 0.12, \sigma = 0.4, r = 0.07, \varepsilon = 1$, the investor using power utility function with parameter $\gamma = 0.5$ will keep all his money in the risky asset. The relative risk aversion of an investor using logarithmic utility is 1, while for the γ-power utility it is $1-\gamma$, that is 0.5 for this case.

3.4 Merton-strategy for the Modified Model

In this section we take a look at how the optimal strategy obtained by using the Black-Scholes model works for the generalized model.

This case resembles an investor who wants to use the Merton-strategy although he knows that the Black-Scholes model is not accurate.

3.4.1 Logarithmic utility

Table below contains the theoretically obtained values for the Merton ratio (MR) and expected utility (OV - optimal value) of this trading strategy in the case of Black-Scholes model and logarithmic utility function, and the expected utility (OV) given by the same trading strategy for the modified model. For these results we took $N = 1000$ discretization steps, $M = 1000000$, the initial wealth is 500, the initial stock price $S_0 = 100$.

The expected utility of the final wealth for the Black-Scholes model denoted by OV, and respectively the optimal policy (MR) are calculated by the formulas:

$$\mathbb{E}[\log(X_T)] = \log(V_0) + (r + \frac{\mu - r}{2\sigma^2})$$

$$u_t^* = \frac{\mu - r}{\sigma^2}$$

Model	μ	σ	r	ε	ρ	MR	OV	Sim OV
BS	0.12	0.4	0.07	-	-	0.3125	6.4409	6.2923
Fads	0.12	0.4	0.07	0.01	0.3	-	-	6.292
Fads	0.12	0.4	0.07	0.1	0.3	-	-	6.2922
Fads	0.12	0.4	0.07	1	0.3	-	-	6.2923
BS	0.5	1	0.1	-	-	0.4	6.5146	6.3952
Fads	0.5	1	0.1	0.01	0.3	-	-	6.3943
Fads	0.5	1	0.1	0.1	0.3	-	-	6.3938
Fads	0.5	1	0.1	1	0.3	-	-	6.3935

We have already seen in the previous section that the best constant allocation strategy for the logarithmic utility does not always agree with the Merton-ratio, and we could not get bigger expected utility for any constant allocation strategy in the case of Fads-models than what we got for the Black-Sholes model, using the Merton-ratio. According to this, the Merton-ratio for the modified model gives slightly worse expected utility for the final wealth than in the Black-Scholes case.

3.4.2 Power utility

Table below contains the theoretically obtained values for the Merton ratio (MR) and expected utility (OV - optimal value) of this trading strategy in the case of Black-Scholes model and power utility function with exponent γ, and the expected utility (OV) of the final wealth, given by the same trading strategy for the modified model. For these results we took $N = 1000$ discretization steps, $M = 1000000$, μ denotes the drift, σ the volatility of the processes, $r = 0$, the initial wealth is set to be 500 and the initial stock price $S_0 = 100$.

The expected utility of the final wealth for the Black-Scholes model denoted by OV, and respectively the optimal policy (MR) are calculated by the formulas:

$$\mathbb{E}\Big[\frac{X_T^\gamma}{\gamma}\Big] = \frac{V_0^\gamma}{\gamma} \exp\Big(\frac{\mu^2\gamma}{2\sigma^2(1-\gamma)}\Big)$$

$$u_t^* = \frac{\mu}{\sigma^2(1-\gamma)}$$

Model	μ	σ	ε	γ	MR	OV	Sim. OV
BS	0.12	0.4	-	0.1	0.8333	18.7098	18.6166
Fads	0.12	0.4	0.01	0.1	-	-	18.7091
Fads	0.12	0.4	1	0.1	-	-	18.7103
Fads	0.12	0.4	100	0.1	-	-	18.7204
BS	0.12	0.4	-	0.01	0.7576	106.4602	106.4585
Fads	0.12	0.4	0.01	0.01	-	-	106.4591
Fads	0.12	0.4	0.1	0.01	-	-	106.4597
Fads	0.12	0.4	1	0.01	-	-	106.4601
BS	0.3	1	-	0.5	0.6	46.7798	46.2725
Fads	0.3	1	0.01	0.5	-	-	46.2643
Fads	0.3	1	0.1	0.5	-	-	46.2882
Fads	0.3	1	1	0.5	-	-	47.5763
BS	0.3	1	-	0.1	0.3333	18.7098	18.6919
Fads	0.3	1	0.01	0.1	-	-	18.7082
Fads	0.3	1	0.1	0.1	-	-	18.7102
Fads	0.3	1	1	0.1	-	-	18.7143

Looking at the results we can see that for $\gamma = 0.01$, when we are "very close" to the case of logarithmic utility, the expected utility of the final wealth decreases as we modify the model, but for the less risk-averse investors, for $\gamma = 0.1$ and $\gamma = 0.5$, the investor can actually profit from the uncertainty in the drift and get even better expected utility.

Conclusion and Further Extensions

In this thesis we studied Merton's classical portfolio problem. First we derived theoretically the optimal allocation strategy using the methods of stochastic control theory, then we tested this result in practice in discrete time simulation scenario. The answers for the questions formulated in the introduction would be:

- Does discretization decrease the expected utility of the final wealth?
  Yes it does. The maximal value obtained theoretically requires continuous trading, a discrete approximation of continuous trading leads to a lower estimate for the utility.

- Will an investor who wants to use a Merton-type of strategy, i.e. a constant allocation strategy, get the same constant if we change the model a little bit?
  For the case when we modified the model just a little bit (i.e. $\varepsilon = 0.01$ in the Fads model), we got the same/almost the same constant allocation strategy as in the Black-Scholes model.

- Is it better to invest less in the risky asset as uncertainty in the drift of the modified model grows or can an investor profit from it?
  It is also up to the risk aversion of the investor: we have seen that the investor using logarithmic utility function is more careful: is this case is better to slightly decrease the amount of money invested in the risky asset, however, less risk-averse investors can get higher expected utility by investing more in stock.

- If we change the model but use the allocation strategy which is optimal in the Black-Scholes case, can we still reach the same expected utility?
  We can reach it or at least we can get close to it. In the logarithmic case, we cannot reach it, however, since the optimal constant allocation strategy for the modified model is very close to the Merton-ratio, the expected utility almost agrees. For the power-utility, we can reach the same expected utility,

but in this case we have seen, that even higher expected utility is possible using another constant allocation strategy.

There are several ways how this problem can be generalized for further study and research:

- Non-random/random income can be added.
- Transaction costs can be added (e.g. proportional or small fixed transaction costs) [1].
- Other types of utility function can be used.
- A more general market model can be used (e.g. stochastic variance, stochastic drift).
- Bankruptcy can be treated [7].

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Appendix

MATLAB code

```
function BlackScholes = BlackScholes(N,M,mu,sigma,T,S0)
    %Construction of Black-Scholes trajectories
    %N discretization steps, M trajectories, mu drift,
    %sigma volatility, T time horizon, S0 initial wealth
    dt = T/N;
    z = randn(N,M);
    BS = (mu-0.5*sigma^2)*dt+sigma*sqrt(dt)*z;
    BS = S0*exp(cumsum(BS));
    BlackScholes = [S0*ones(1,M);BS];
end

function Fads = Fads(N,M,epsilon,sigma,mu,ro,T,S0)
    %Construction of Fads trajectories
    %N discretization steps, M trajectories, mu drift,
    %sigma volatility, T time horizon, S0 initial wealth
    %epsilon constant for the O-U process
    %ro correlation of Brownian motions

    %simulation of O-U process
    dt = T/N;
    t = 0:dt:T;
    w1 = randn(N,M);
    w2 = randn(N,M);
    c = exp(-1/epsilon.*t);
    OU = zeros(N+1,M);
    for m = 1:M
        OU(:,m) = diag(c)*cumsum(diag(exp(1/epsilon*t))*
        [0; sqrt(dt)*w1(:,m)]);
```

```
    end

    %simulation of FADS model
    St = zeros(N+1,M);
    St(1,:) = S0*ones(M,1);
    for m = 1:M
        for n = 1:N
            St(n+1,m) = St(n,m) + St(n,m)*((mu+OU(n,m))*dt+
            sigma*ro*sqrt(dt)*w1(n,m)+sigma*sqrt(1-ro^2)*
            sqrt(dt)*w2(n,m));
        end
    end
    Fads = St;
end

function mert = mert(S,mr,V0,N,t,r)
    %returns the final wealth, when the stock prices
    %follow the process S, mr Merton ratio,
    %r interest rate, V0 initial wealth
    B = exp(r.*t); %value of bond at time t
    %(1-mr)*V0 wealth in bond at 0,
    %mr*V0 wealth in stock at time 0
    NB = (1-mr)*V0/B(1).*ones(1,N+1);
    %init number of bonds held at i
    NS = mr*V0/S(1).*ones(1,N+1);
    %init number of stocks held at i
    for i = 1:N
        wealth = NB(i)*B(i+1)+NS(i)*S(i+1);
        %total wealth ar time i+1
        NB(i+1) = (1-mr)*wealth/B(i+1);
        %rebalancing
        NS(i+1) = mr*wealth/S(i+1);
        %rebalancing
    end
    mert = NB(N)*B(N)+NS(N)*S(N);
end
```

```
function [ov,op] = tvlog(mu,sigma,r,V0)
    %Calculating the true maximal expected utility (ov)
    %and the Merton-ratio (op) for log utility
    beta = (mu - r)/sigma;
    op = beta/sigma;
    ov = log(V0)+(r + (mu - r)/(2*sigma^2));
end

function [ov,op] = tvpow(mu,sigma,r,V0,alpha)
    %True values for the maximal expected utility (ov)
    %and Merton-ratio (op) for power utility
    op = (mu-r)/((1 - alpha)*sigma^2);
    cons = mu^2*alpha/(2*sigma^2*(1-alpha));
    ov = ((V0^alpha)/alpha)*exp(cons);
end

function ExpUtLog = ExpUtLog(S,mu,sigma,M,N,V0,t,r)
    %Computing the expected utility of final wealth using
    %the strategy given by the Metron-ratio achieved
    %theoretically for the FADS-model for log utility
    [ov,op] = tvlog(mu,sigma,r,V0);
    for m = 1:M
        x(m) = mert(S(:,m),op,V0,N,t,r);
    end
    ExpUtLog = mean(log(x));
end

function ExpUtPow = ExpUtPow(S,mu,sigma,M,N,V0,t,r,alpha)
    %Computing the expected utility of final wealth using
    %the strategy given by the Metron-ratio achieved
    %theoretically for the FADS-model for power utility
    [ov,op] = tvpow(mu,sigma,r,V0,alpha);
    for m = 1:M
        x(m) = mert(S(:,m),op,V0,N,t,r);
    end
    ExpUtPow = mean((x.^alpha)/alpha);
end
```

```
function [val,opt] = OptConstLog(S,N,M,V0,t,r)
    %Finds the optimal constant allocation strategy
    %S stock price trajectories, N discretization steps, M
    %trajectories simulated, V0 initial wealth, t time, r
    %interest rate
    ut = zeros(1000,1);
    for i = 1:1000
        x = zeros(M,1);
        for m = 1:M
            x(m) = mert(S(:,m),i/1000,V0,N,t,r);
        end
        ut(i) = mean(log(x));
    end
    [val,opt] = max(ut);
end

function [val,opt] = OptConstPow(S,N,M,V0,t,r,alpha)
        %Finds the optimal constant allocation strategy and
        %the optimal value
        %S stock price trajectories, N discretization steps,
        %M trajectories simulated, V0 initial wealth, t time,
        %r interest rate

        ut = zeros(1000,1);
        for i = 1:1000
        x = zeros(M,1);
        for m = 1:M
                x(m) = mert(S(:,m),i/1000,V0,N,t,r);
        end
    ut(i) = mean((x.^alpha)/alpha);
        end
        [val,opt] = max(ut);
end
```